Ramsey theoretic notions of convergence
Lecture 3 - Countable Compactness on Barriers

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Various convergence notions

For an ideal $\mathcal{I}$ on ω, the Fubini power $B^{\mathcal{I}}$ is an ideal on B defined recursively on the rank of B by

$$B^{\mathcal{I}} = \{A \subseteq B : \{n : A[n] \notin \mathcal{I}\} \in \mathcal{I}\}$$

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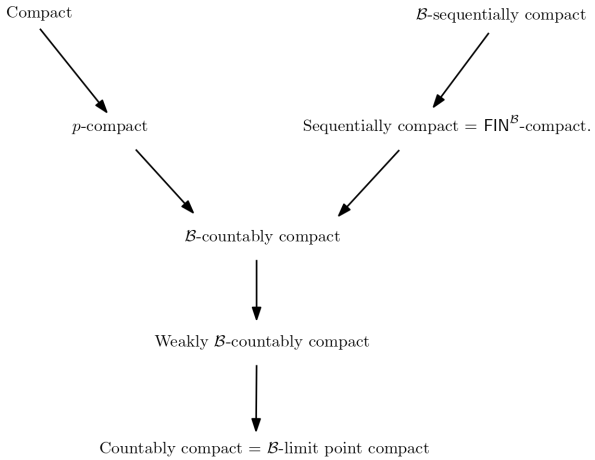
Definitions

Given a barrier B and a space X we say

1. X is B-countably compact if every $f : B \to X$ has a B^p limit for some p.
2. X is weakly B-countably compact if every $f : B \to X$ has a B-accumulation point.

Basic implications



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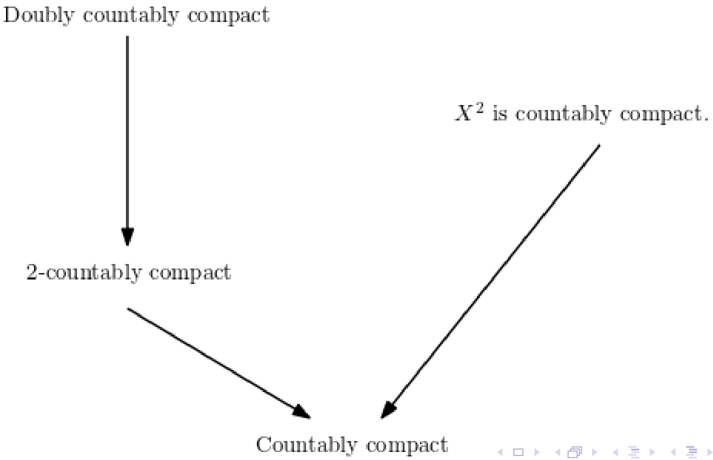

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There is a $< \omega$-countably compact $X \subseteq \beta\omega$ such that X^2 is not countably compact but we don't know if there is a B-countably compact example even for $B =$ the Schreier barrier

A 2-countably compact Z whose square is not countably compact

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(2) If $Z \subseteq \omega^*$ has size $< 2^{2^{\aleph_0}}$ then $\beta\omega \setminus Z$ is countably compact.

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Corollary

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The Hindman-Carlson-Strauss Theorem fails for the Schreier barrier.

Proposition

Let B be the Schreier barrier. Then there exists a countable LOTS L and a function $f : B \to L$ such that $f[B|_M]$ is never discrete for M infinite.

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Example (Hrusak, van Mill, Ramos-Garcia, Shelah)

There is a ZFC example of a countably compact group without convergent sequences, hence countably compact groups with product not countably compact.

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Lets do the special case $n = 2$

Theorem

If G is a topological group that is torrent 2-countably compact, then G^2 is countably compact.

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The example of [Hrušák, van Mill, Ramos-Garcia and Shelah] can be modified to produce

1. a topological group G that is B-countably compact (for every barrier B) which has no nontrivial convergent sequences and whose square is not countably compact (so is not 2-torrent countably compact).
2. a topological group H that is torrent B-countably compact (for every barrier B) with no nontrivial convergent sequences (and so H^ω is countably compact)

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