Universally Measurable Sets III

Paul Larson

Department of Mathematics
Miami University
Oxford, Ohio 45056

larsonpb@miamioh.edu

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Borel reinterpretations (again)

Recall from last time:

Definition. Given a subset A of a Polish space X, a partial order $\mathbb{P}$ and a V-generic filter G, the *Borel reinterpretation* of A in $V[G]$ is the union of all the (reinterpreted) ground model Borel sets contained in A.

Proposition. A subset A of a Polish space X is universally measurable if and only if, whenever $V[G]$ is an extension of V via random forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

Proposition. A subset A of a Polish space X is universally categorical if and only if, whenever $V[G]$ is an extension of V via Cohen forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

Tree reinterpretations

A *tree* on X is a set of finite sequences from X closed under initial segments. We write $[T]$ for the set of infinite branches through a tree T.

Given a tree T on $X \times Y$, the *projection* of T ($\mathrm{p}[T]$) is the set of $x \in X^\omega$ for which there is a $y \in Y^\omega$ with $(x, y) \in [T]$.

Given a subset A of a set of the form X^ω, the *tree-reinterpretation* of A in a generic extension is the union of all sets of the form $\mathrm{p}[T]$, where T is a tree in the ground model with $\mathrm{p}[T] \subseteq A$ in the ground model.

This is the same as the Borel reinterpretation in the case where the extension is via a c.c.c. forcing.

universally Baire sets

Definition. Given a partial order $\mathbb{P}$ and a set X, a set $A \subseteq X^\omega$ is $\mathbb{P}$-*Baire* if, in any forcing extension by $\mathbb{P}$, the tree-reinterpretations of A and $X^\omega \setminus A$ are complements.

Using this definition, universal measurability (for subsets of 2^ω or ω^ω) is random-Baireness and universal categoricity is Cohen-Baireness.

If $\mathbb{P}$ regularly embeds into $\mathbb{Q}$ then $\mathbb{Q}$-Baire implies $\mathbb{P}$-Baire.

A set is said to be *universally Baire* if it is $\mathbb{P}$-Baire for all partial orders $\mathbb{P}$.

Thin uB sets

A subset of a Polish space is said to be *thin* if it does not contain a perfect set.

If A is a thin $\mathbb{P}$-Baire subset of ω^ω, then every element of ω^ω appearing in any generic extension by $\mathbb{P}$ is either in A or in $\mathrm{p}[T]$ for some T in V with $\mathrm{p}[T] \cap A = \emptyset$.

So: thin universally measurable sets are the same as universally null sets and thin universally categorical set are the same as universally meager sets.

Question

(Kumar) Does ZFC prove that there is a universally measurable set which is not universally Baire? (ZFC + a Woodin cardinal does)

Two special cases:

- Can every universally null set be Hechler-Baire?
- Can every universally measurable subsets of $(2^\omega)^\omega$ be $\mathrm{Col}(\omega, 2^\omega)$-Baire?

More questions

Is it consistent with ZFC that every universally measurable set has universally null symmetric difference with some universally Baire set?

Is it consistent with ZFC that the universally measurable sets are the members of the smallest σ-algebra containing the universally Baire sets and the universally null sets (and closed under the Suslin operation)?

Is it consistent with ZFC that there exists a thin universally Baire set of cardinality $\aleph_2$ (yes for $\aleph_1$, but no even for $\aleph_1$ if there exists a Woodin cardinal)?

Polar ideals

Joint work with Jindřich Zapletal:

Definition. Let I be a σ-ideal on a Polish space X which is generated by analytic sets. We say that I is *polar* if there is a set $M \subseteq \mathrm{Meas}(X)$ such that

$$I = \{A \subseteq X : \forall \mu \in M\, \mu(A) = 0\}.$$

We say that I is $\mathbf{\Sigma}^1_1$-*polar* if there exists an analytic $M \subseteq \mathrm{Meas}(X)$ witnessing that I is polar.

If the quotient forcing $P_I = \mathrm{Borel}(X)/I$ is proper in all forcing extensions, then we say that I is an *iterable* $\mathbf{\Sigma}^1_1$-polar ideal, and we call P_I a *polar forcing*.

Examples of polar forcings

Random forcing (the Lebesgue null ideal)

Sacks forcing (the ideal of countable sets, induced by letting $M = \mathrm{Meas}(X)$)

The countable product of Sacks forcing. This is equivalent to $\mathrm{Borel}((2^\omega)^\omega)/I$, where I is the ideal of subsets of $(2^\omega)^\omega$ not containing a product of perfect sets. M is the set of product measures concentrating on a perfect szet in each coordinate.

The Borel sets modulo the ideal generated by the analytic $\mathbb{E}_0$-selectors.

Iterations

Theorem (Larson-Zapletal) A countable iteration of polar forcings is polar.

The successor step

Given $B \subseteq X_0 \times X_1$, and $x \in X_0$, $B_x = \{y : (x, y) \in B)\}$.

Definition. Suppose that, for each $i \in \{0, 1\}$, $M_i \subseteq \mathrm{Meas}(X_i)$, for some Polish space X_i. We define $M_0 * M_1$ to be the set of measures on $X_0 \times X_1$ of the form

$$\nu(B) = \int f(x)(B_x)\, d\mu,$$

where $\mu \in M_0$ and $f\colon X_0 \to M_1$ is Borel.

Lemma. If each I_i is an iterable $\mathbf{\Sigma}^1_1$-polar ideal, then $M_0 * M_1$ an analytic set of measures giving rise to an iterable $\mathbf{\Sigma}^1_1$-polar ideal $I_0 * I_1$ on $X_0 \times X_1$. Moreover, $P_{I_0} * P_{I_1}$ is forcing-equivalent to $P_{I_0 * I_1}$.

The limit step

The $*$ operation on sets of measures is associative.

Given $M_i \subseteq \mathrm{Meas}(X_i)$ $(i \in \omega)$ we let $*_i M_i$ be the set of $\mu \in \mathrm{Meas}(\prod_{i\in\omega} X_i)$ such that, for each $j \in \omega$, the preimage measure induced by the projection of $\prod_{i\in\omega} X_i$ to $\prod_{i\leq j} X_i$ and μ is in $*_{i\leq j} M_i$.

Then $*_{i\in\omega} M_i$ an analytic set of measures giving rise to an iterable $\mathbf{\Sigma}^1_1$-polar ideal $*_{i\in\omega} I_i$ on $\prod_{i\in\omega} X_i$.

Moreover, the full-support iteration of the partial orders P_{I_i} is forcing-equivalent to $P_{*_{i\in\omega} I_i}$.

Measured extensions (I)

Definition. Let $M \subseteq N$ be transitive models of set theory with the same ordinals. We say that N is a *measured extension* of M if the following hold.

- Every $a \subseteq M$ which is a countable set in N is a subset of a $b \in M$ which is countable in M.
- Suppose that $\epsilon \in \mathbb{Q}^+$, X is a Polish space in M, and F is, in M, a collection of Borel subsets of X. If there is a Borel probability measure μ on X in the model N such that for every $B \in F$, $\mu(B) \geq \epsilon$, then there is a Borel probability measure ν in the model M such that for every $B \in F$, $\nu(B) \geq \epsilon$.

Measured extensions (II)

Measured extensions are ω^ω-bounding and preserve outer measure, and the relation of being a measured extension is transitive.

If N is a measured extension of M, and A is, in M, a universally measurable subset of a Polish space X, then the Borel reinterpretations of A and $X \setminus A$ are complements in N.

Moreover, then Borel reinterpretation of A is universally measurable in N.

Theorem. (Larson-Zapletal) Let I be an iterable Σ^1_1-polar ideal on a Polish space X. Any forcing extension by a countable support iteration of the partial order P_I is measured.

Proof sketch

We sketch the proof for a single forcing P_I, which gives the theorem for countable iterations by the iterability of polar ideels. The general case follows from standard forcing arguments.

Suppose that

- X is a Polish space,
- F is a set of Borel subsets of X,
- $\epsilon \in \mathbb{Q}^+$,
- p is a P_I-condition and
- τ is a P_I-name for a measure on X such that, for all $B \in F$, $p \Vdash \tau(B) \geq \epsilon$.

We may also fix a Borel function $f : p \to \mathrm{Meas}(X)$ such that $p \Vdash \tau = f(g)$.

Fix a probability measure ν on X such that $\nu(p) = 1$ and all analytic sets in I have ν-measure 0.

Let μ be the Borel probability measure on X defined by

$$\mu(B) = \int f(x)(B) d\nu(x).$$

Suppose towards a contradiction that $\mu(B) < \epsilon$ for some $B \in F$. Then

$$r = \{x \in p : f(x)(B) < \epsilon\}$$

has positive ν-measure and r forces that

$$f(g)(B) = \tau(B) < \epsilon,$$

giving a contradiction. $\square$

Projective sets

Recall that a set is $\mathbf{\Sigma}^1_1$ it is a continuous image of a Borel set, and $\mathbf{\Pi}^1_1$ if it is the compplement of a $\mathbf{\Sigma}^1_1$ set.

More generally, a set is

- $\mathbf{\Sigma}^1_{n+1}$ if it is a continuous image of a $\mathbf{\Pi}^1_n$ set,
- $\mathbf{\Pi}^1_{n+1}$ if it is the complement of a $\mathbf{\Sigma}^1_{n+1}$ set and
- $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Each class $\mathbf{\Sigma}^1_{n+1} \setminus \mathbf{\Delta}^1_n$ is nonempty.

The *projective sets* are the members of $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_1$.

The influence of large cardinals

If there exist infinitely many Woodin cardinals, then every projective set is universally measurable and universally categorical (and moreover, $\mathbb{P}$-Baire for all $\mathbb{P}$ of cardinality less than the supremum of the Woodin cardinals).

If there exists one Woodin cardinal, then no universally null set of cardinality $\aleph_1$ is universally Baire.

If there exist proper class many Woodin cardinals, then the universally Baire sets are exactly those have a definition φ (with parameters) such that, whenever M and N are generic extensions of V, and $x \in M \cap N$, $M \models \varphi(x)$ if and only if $N \models \varphi(x)$.

Theorem. (Larson-Shelah) If there exists an $r \subseteq \omega$ such that $V = L[r]$, then there is a measured proper forcing extension in which every universally measurable set is $\mathbf{\Delta}^1_2$ and every universally categorical set is $\mathbf{\Delta}^1_2$.

Proof strategy

The idea of the proof is : iterate to make

$$[\omega^\omega]^{\aleph_1} \subseteq \boldsymbol{\Sigma}^1_2$$

and everthing will work itself out.

The hypothesis "$V = L[r]$ for some $r \subseteq \omega$" is used to produce an absolutely $\Delta^1_2(r)$ set of reals coding a ladder system

$$\bar{C} = \langle C_\alpha : \alpha < \omega_1 \rangle$$

on ω_1 (a choice of a cofinal subset of ordertype ω for each countable limit ordinal).

Pathological functions

Given an ideal I on 2^ω, we say that a partial function

$$F\colon 2^\omega \to 2$$

is *I-pathological* if, whenever

- $s \in 2^{<\omega}$,
- $i < 2$ and
- $B \subseteq [s]$ is Borel and I-positive,

there is a $x \in B \cap [s] \cap \mathrm{dom}(F)$ such that and $F(x) = i$.

We say *null-pathological* when I is the ideal of Lebesgue-null sets, and *totally pathological* when I is the ideal of countable sets.

ZFC implies the existence of totally pathological functions, and if $r \subseteq \omega$ is such that $\mathbf{V} = \mathbf{L}[a]$, then there exists such a function F which is $\Delta^1_2(r)$.

The iteration

The forcing construction is a countable support iteration of partial orders $Q_{\bar{C},F,g}$ which produce (by countable initial segments), given

$$g\colon \omega_1 \to 2,$$

a function

$$h\colon \omega_1 \to 2$$

such that, for all countable limit ordinals α,

$$F(h{\restriction}C_\alpha) = g(\alpha).$$

The partial orders $Q_{\bar{C},F,g}$ are proper and do not add reals.

However, for each $E \in [\omega^\omega]^{\aleph_1}$ there is a length-ω iteration of partial orders of the form $Q_{\bar{C},F,g}$ producing a subset of $\omega \times \omega$ coding E relative to $\bar{C}$ and F, and thereby making X $\mathbf{\Sigma}^1_2$. .

Everything works itself out

As in the previous forcing constructions, one uses the fact that, for every universally measurable set $A \subseteq 2^\omega$ in the forcing extension $V[G]$ (after the iteration), there is an intermediate extension $V[G_\beta]$ in which $A \cap V[G_\beta]$ is universally measurable, and that for each Borel set $B \in V[G_\beta]$,

$$A \cap B \cap V[G_\beta] = \emptyset$$

implies

$$A \cap B = \emptyset$$

As before, these Borel sets determine A.

Local randomness (I)

Fix a condition p in our iteration P, a P-name τ for an element of 2^ω and a countable elementary submodel M of a large enough set with $P, p, \tau \in M$.

We build a finitely branching tree T of height ω consisting of conditions in $P \cap M$ below p, where the conditions on the nth level are in the nth dense subset of P in M.

We then get a Borel function $f : [T] \to 2^\omega$, where $f(x)$ is the realization of τ according to the conditions in x.

Local randomness (II)

T will have the property that for any Borel set $E \subseteq [T]$ of positive Lebesgue measure, some $q \leq p$ forces that the generic filter will contain an M-generic element of E.

Using the universal measurability of $A \cap V[G_\beta]$, we can find a q forcing that the realization of τ will be in a Borel set from $V[G_\beta]$ which is ether contained in or disjoint from $A \cap V[G_\beta]$.

Preserving pathology

To make this argument work, we need that the function F remains null-pathological throughout the iteration.

This follows from the preservation of Lebesgue outer measure under countable support iterations of proper forcings not adding reals.

In the resulting model, every universally measurable set is a union of $\aleph_1$ many Borel sets (in addition to being $\boldsymbol{\Delta}^1_2$) and so is every universally categorical set.

However, there are universally null sets which are not meager, and universally meager sets which are not Lebesgue null.

The forcing extension is measured (so universally measurable sets reinterpret as universally measurable sets), although it is not evidently induced by polar forcings.

Medial Limits

A *medial limit* is a universally measurable finitely-additive atomless probablility measure on $\mathcal{P}(\omega)$.

So, a medial limit is a function $m\colon \mathcal{P}(\omega) \to [0,1]$ such that

- $m(\omega) = 1$;
- $m(a) = 0$ for all finite $a \subseteq \omega$;
- $m(A + B) = m(A) + m(B)$ whenever $A, B \subseteq \omega$ are disjoint;
- for all borel $E \subseteq [0,1]$, $m^{-1}[E]$ is universally measurable.

ZFC does not prove that medial limits exist (Larson), but ZFC + Martin's Axiom does (Christensen, Mokododzki, Normann).

An application to equivalence relations

Theorem (Kechris) Assume that there exists a medial limit. Let E be an equivalence relation induced by a Borel action of a countable amenable group G on a Polish space X. Then there exist a hyperfinite Borel equivalence relation F on a Polish space Y and a universally measurable isomorphism $f\colon X \to Y$ of E with F.

RCC

Sketch of a proof of the existence of medial limits from Martin's Axiom:

For each $i \in \omega$, we let δ_i be the associated indicator function on $\mathcal{P}(\omega)$. That is, $\delta_i(x)$ is 1 if $i \in x$ and 0 otherwise.

A *rational convex combination* of functions $f_1, \ldots, f_n$ from some fixed set X into $\mathbb{R}$ is a function (on X) of the form

$$\nu(x) = q_1 f_1(x) + \cdots + q_n f_n(x)$$

for nonnegative rational numbers $q_1, \ldots, q_n$ summing to 1.

We let RCC be the set of rational convex combinations of the functions δ_i $(i \in \omega)$.

SRC

We let SRC be the set of sequences p from RCC whose supports are increasing.

A set $p \in \mathrm{SRC}$ is naturally enumerated as $\langle \nu^p_n : n \in \omega \rangle$.

f_p

Given a $p \in \mathrm{SRC}$, we let $f_p \colon \mathcal{P}(\omega) \to [0,1]$ be the partial function defined by setting

$$f_p(x)$$

to be $\lim_{n\to\infty} \nu^p_n(x)$ when this limit exists.

Each such f_p is finitely additive, and takes value 1 on ω and 0 on finite sets.

The domain of each f_p is meager, however.

$\leq_0$

We define $\leq_0$ on SRC by setting

$$p \leq_0 q$$

to hold if $\forall^\infty n \; \nu^p_n \in \mathrm{RCC}(q)$.

Then:

- $p \leq_0 q$ implies that $f_q \subseteq f_p$;
- every countable $\leq_0$-decreasing sequence has a lower bound.

Mokobodzki's Lemma

The following lemma is the key technical tool for building medial limits.

Lemma (Mokobodzki) For all $p \in \mathrm{SRC}$ and $\mu \in \mathrm{Prob}(\mathcal{P}(\omega))$, then there exists a

$$q \leq_0 p$$

such that

$$\mu(\mathrm{dom}(f_q)) = 1.$$

Using the lemma, and Martin's Axiom to find a lower bound for each $\leq_0$-descending sequence of length less than the continuum, one can build a descending $\leq_0$-sequence

$$\langle p_\alpha : \alpha < \mathfrak{c} \rangle$$

such that for each $\mu \in \mathrm{Prob}(\mathcal{P}(\omega))$ there is an α such that $\mu(\mathrm{dom}(f_{p_\alpha})) = 1$.

Then $\bigcup_{\alpha<\omega_1} f_{p_\alpha}$ is a medial limit.

A universally measurable ideal without the Baire Property

If m is a medial limit, then the set $I = m^{-1}[\{0\}]$ is a universally measurable ideal.

I cannot have the property of Baire, howeever.

To see this, note first that I is invariant under finite changes, so if it has the property of Baire then it is either meager or comeager.

If it is comeager, then there exists an $A \in I$ with $\omega \setminus A \in I$.

If it is meager, then $\mathcal{P}(\omega) \setminus I$ contains a perfect set P whose elements have finite intersection. Then for for some $\epsilon > 0$ the set $\{x \in P : m(X) > \epsilon\}$ is uncountable.

The Filter Dichotomy

Definition. *The Filter Dichotomy* is the statement that for each nonmeager filter F on ω, there is a finite-to-one function $h\colon \omega \to \omega$ such that $\{h[x] | x \in F\}$ is an ultrafilter.

Blass and Laflamme showed that the Filter Dichotomy holds in the Miller model and another model previously considered by Blass and Shelah.

Theorem (Larson) The Filter Dichotomy implies that universally measurable uniform filters on ω are meager.

Proof. Let F be a nonmeager universally measurable uniform filter on ω, and let $h\colon \omega \to \omega$ be finite-to-one such that $\{h[x] : x \in F\}$ is an ultrafilter. Let

$$S = \{\bigcup_{n \in Z} h^{-1}[\{n\}] : Z \subseteq \omega\},$$

and let $G\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be defined by $G(x) = h[x]$. Then:

- S is a perfect subset of $\mathcal{P}(\omega)$.
- $F \cap S$ is a universally measurable subset of S.
- $G \restriction S\colon S \to \mathcal{P}(\omega)$ is a homeomorphism.
- $G[F \cap S] = G[F]$ is a nonprincipal ultrafilter, so not Lebesgue measurable.

This gives a contradiction.

Brendle and I used almost the same argument to show that ultrafilter limits of asymptotic density do not give universally measurable functions.

Semifilters

A semifilter on ω is a proper subset of $\mathcal{P}(\omega)$ which is closed under finite changes and supersets.

The following statement was shown by Laflamme to hold in the models of the Filter Dichotomy mentioned above.

Definition. The Semifilter Trichotomy is the statement that for every semifilter F on ω there is a finite-to-one function $h\colon \omega \to \omega$ such that $\{h[x] : x \in F\}$ is either the cofinite filter, a nonprincipal ultrafilter or the set of all infinite subsets of ω.

The Semifilter Trichotomy is equivalent to $\mathfrak{u} < \mathfrak{g}$.

The Filter Dichotomy argument above can be adapted to show the following.

Theorem (Larson) The Semifilter Trichotomy implies that universally measurable semifilters on ω have the property of Baire.

Question

Do medial limits exist in the Laver model?

Question

Let M be the set of σ-additive complete probability measues μ on $\mathcal{P}(\omega)$ for which

$$\mu(\{x \subseteq \omega : n \in x\}) \to 0$$

and let N be the set of σ-additive complete measues μ on $\mathcal{P}(\omega)$ for which

$$\mu(\{x \subseteq \omega : n \in x\}) \to 1.$$

Must there be a universally measurable set (or any set) which has measure 0 for all the measures in M and measure 1 for all the measures in N?

Yes if there is a medial limit.

Such a set cannot have the Baire property.

Reinterpreting medial limits

If N is a measured extension of M, and m is a medial limit in N (more generally, if the Borel reinterpretions in N of any universally measurable set in M and its complement are complements in N), then m induces a medial limit m' in N.

This essentially follows from the fact that, for each $r \in [0, 1] \cap M$, the Borel reinterpretations of the sets $m^{-1}[0, r)$ and $m^{-1}[r, 1]$ are complements in N.

Question

Does the existence of a finitely additive atomless probability measure on $\mathcal{P}(\omega)$ (equivalently, a medial limit) imply the existence of a nonmeasurable set (or an utrafilter on ω)?

Forcing to add a generic medial limit over a model of ZF + DC using the order $\leq_0$ does add a nonprincipal ultrafilter.

Questions

Is being uniformly universally measurable an interesting property?

Are there interesting cardinal characteristics associated to absolute continuity?

Can $\mathbb{P}_{\max}$ be used to prove interesting things about the universally measurable sets?

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