Universally Measurable Sets II

Paul Larson

Department of Mathematics
Miami University
Oxford, Ohio 45056

larsonpb@miamioh.edu

February 3, 2026

At the end last time we produced a nonmeager universally null set (so a universally measurable set without the Baire property) from the assumption $\mathfrak{d} = \mathrm{cof}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N})$.

We will soon see that in the random model every universally null set is universally meager.

First we will show that $\mathrm{add}(\mathcal{N}) = \max\{\mathfrak{b}, \mathfrak{s}\}$ also implies that there is a universally measurable set without the Baire property.

$\mathbb{G}_0$ graphs

For $D \subseteq 2^{<\omega}$, let $G(D)$ be the graph on 2^ω connecting x and y if they disagree in exactly one place, and their longest comment initial segment is in D.

A $\mathbb{G}_0$-graph is a graph of the form $G(D)$ when D is dense and has exactly one member of length n for each $n \in \omega$. We let $\mathbb{G}_0$ be the set of $\mathbb{G}_0$-graphs.

The $\mathbb{G}_0$-dichotomy (Solecki-Kechris-Todorceivic) says that if $G \in \mathbb{G}_0$ and H is an analytic graph on a Polish space, then either H has a Borel $\mathbb{N}$-coloring or there is a homomorphism from G to H.

More on $\mathbb{G}_0$-graphs

Each $G \in \mathbb{G}_0$ is connected, acyclic and locally countable.

The connected components are the $\mathbb{E}_0$-degrees.

Comeagerly many $x \in 2^\omega$ lie in a connected component of G whose members all have infinite degree.

Measure 1 many $x \in 2^\omega$ have finite degree in G.

Baire colorings

Given a graph G on a Polish space X and a collection of sets Γ, let $\chi(G, \Gamma)$ be the smallest cardinality of a set of G-independent sets from Γ whose union contains X.

If $G \in \mathbb{G}_0$ and $A \subseteq 2^\omega$ is G-independent, then for no $s \in 2^{<\omega}$ is

$$A \cap [s]$$

comeager in s.

Letting Δ_{B} be the collection of subsets of 2^ω with the Baire property, it follows that

$$\chi(G, \Delta_{\mathrm{B}}) \geq \mathrm{cov}(\mathcal{M}).$$

Measurable colorings

Let Δ_{U} be the set of universally measurable subsets of 2^ω.

Ben Miller has shown that $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ implies that, for any $G \in \mathbb{G}_0$,

$$\chi(G, \Delta_{\mathrm{U}}) \leq 3.$$

(Gaspar-Larson) The argument goes through assuming

$$\mathrm{add}(\mathcal{N}) = \max\{\mathfrak{b}, \mathfrak{s}\}.$$

Recall that the collection of universally measurable sets is closed under unions of cardinality less than $\mathrm{add}(\mathcal{N})$.

It is an open question whether ZFC implies that $\chi(G, \Delta_{\mathrm{U}}) \leq 3$ for any $G \in \mathbb{G}_0$.

Free functions

A (Borel) function $f\colon X \to X$ is said to be *free* if, for all $x \in X$ and $m \in \mathbb{N}$,

$$f^{(m)}(x) \neq x.$$

The graph $\mathcal{G}_f$ connects each $x \in X$ with $f(x)$.

Theorem. $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$ if and only if there exists a universally measurable set $A \subseteq X$ such that for every $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

Given $x \in X$, we call $\{f^i(x) : i \in \omega\}$ the *f-forward image* of x.

The proof of of Miller's theorem involves 3-coloring part of the graph G in a Borel way, and then proving the following lemma, which implies that $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$.

Lemma. (B. Miller) Suppose that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$. Let X be a Polish space, $f\colon X \to X$ be a free Borel functon, and

$$\langle F_n : n \in \omega \rangle$$

be a $\subseteq$-increasing sequence of Borel equivalence relations whose union contains $\mathcal{G}_f$, such that no F_n-class contains any set of the form

$$\{f^i(x) : i \in \omega\}.$$

Then there exists a universally measurable set $A \subseteq X$ such that, for all $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

We will sketch a proof of the lemma from the weaker hypothesis $\mathrm{add}(\mathcal{N}) = \max\{\mathfrak{b}, \mathfrak{s}\}$.

Given x, let $e(x)$ be the least n such that $xF_n f(x)$. For every x,

$$\{e(f^i(x)) : i \in \omega\}$$

is infinite.

It follows that for each Borel probability measure μ there is a function $g_\mu \colon \omega \to \omega$ such that, for each $n \in \omega$,

$$\mu(\{x : \exists i\, e(f^i(x)) \in (n, g_\mu(n)]\}) > 1 - (1/n).$$

Given $A \subseteq \omega$, let D_A be the set of x in X such that

$$\{i : e(f^i(x)) \in A\}$$

and

$$\{i : e(f^i(x)) \notin A\}$$

are both infinite.

Each D_A is Borel and f-invariant.

Lemma. Let μ be a Borel measure on 2^ω and let $A \subseteq \omega$ be such that A contains infinitely many intervals of the form $(n, g_\mu(n)]$, and is also disjoint from infinitely many such intervals. Then $\mu(D_A) = 1$.

$\max\{\mathfrak{b}, \mathfrak{s}\}$

Let γ be the least cardinality of a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that, for every increasing $f : \omega \to \omega$ there exists an $A \in \mathcal{A}$ such that the sets

$$\{n \in \omega : (n, f(n)] \subseteq A\}$$

and

$$\{n \in \omega : (n, f(n)] \cap A = \emptyset\}$$

are both infinite.

Then

$$\gamma = \max\{\mathfrak{b}, \mathfrak{s}\},$$

where $\mathfrak{s}$, the *splitting number*, is the smallest cardinality of a set $\mathcal{B} \subseteq \mathcal{P}(\omega)$ such that, for all infinite $A \subseteq \omega$ there exists $B \in \mathcal{B}$ with $A \cap B$ and $A \setminus B$ infinite.

Fix a family $\mathcal{A} = \langle A_\alpha : \alpha < \gamma \rangle$ witnessing the value of γ.

Each D_{A_α} is Borel, and thus universally measurable.

If $\gamma = \mathrm{add}(\mathrm{null})$, then

$$\bigcup_{\beta<\gamma}(D_{A_\beta} \setminus \bigcup_{\alpha<\beta} D_{A_\alpha})$$

is a universally measurable set as desired.

$\mathfrak{c} = \aleph_2$?

$\mathfrak{s} \leq \text{non}(\mathcal{M}), \mathfrak{d}, \text{non}(\mathcal{L})$.

If $\mathfrak{c} = \aleph_2$ and every universally measurable set has the Baire property, then

- $\aleph_2 = \max\{\mathfrak{b}, \mathfrak{s}\} > \text{add}(\mathcal{N}) = \aleph_1$ (by Miller-Gaspar-Larson) so
- $\aleph_2 = \mathfrak{d} = \text{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ so
- $\aleph_1 = \text{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M})$ (by Brendle-Larson) so
- $\mathfrak{s} = \aleph_1$ and $\mathfrak{b} = \aleph_2$.

This collection of values holds in the Laver and random/Laver models.

Codes

For the rest of this talk we will look at universally measurable sets in generic extensions.

Recall that a Polish space (X, τ) has a countable dense set and is complete with respect to some metric. We can treat the restriction of the metric to the dense set as a code for the space, and use this code to reinterpret the Polish space in generic extensions.

Similarly, each Borel (analytic/projective) subset of X has a definition (using a hereditarily countable parameter) which we can use to reinterpret the set in generic extensions.

Cohen and random (I)

Given an ideal I on a Polish space (X, τ), let P_I be the partial order $\mathrm{Borel}(X)/I$ (without the class for the emptyset), ordered by containment.

- Cohen forcing is P_I for I the set of meager subsets of 2^ω; it is also the partial order $2^{<\omega}$. Cohen forcing adds an element of 2^ω which is not in any (reinterpreted) meager Borel set from the ground model.

- Random forcing is P_I for I the set of Lebesgue-null subsets of 2^ω. Random forcing adds an element of 2^ω which is not in any Lebesgue-null Borel set from the ground model.

Cohen and random (II)

The universally null and universally meager sets can each consistently be contained in the other.

Starting with a model of GCH,

- A finite support iteration of Cohen forcing of length ω_2 forces that

$$\mathrm{UMeag}(2^\omega) \subseteq [2^\omega]^{\leq\aleph_1} \subseteq \mathrm{UNull}(2^\omega).$$

- A finite support iteration of random forcing of length ω_2 forces that

$$\mathrm{UNull}(2^\omega) \subseteq [2^\omega]^{\leq\aleph_1} \subseteq \mathrm{UMeag}(2^\omega).$$

We will sketch the proof over the next few slides.

Proof (I)

Suppose that $V \models$ GCH, and that G is V-generic for a finite-support iteration of Cohen or random forcing of length ω_2.

For each $\beta < \omega_2^V$, let G_β be the restriction of G to the first β stages of the forcing.

Proof (II)

For the second inclusions, note that if A is a set of reals of cardinality at most $\aleph_1$ in $V[G]$, then there is a $\beta < \omega_2^V$ such that $A \subseteq V[G_\beta]$.

Cohen forcing makes the set of ground model reals Lebesgue null, so (in the Cohen iteration case) A is Lebesgue null in $V[G_{\beta+1}]$.

Random forcing makes the set of ground model reals meager, so in the random iteration case A is meager in $V[G_{\beta+1}]$.

Recall that if every set of reals of cardinality at most $\aleph_1$ is Lebesgue null, then every set of reals of cardinality at most $\aleph_1$ is universally null (and similarly for meagerness).

Proof (III)

For the first inclusions, since universal nullness and meagerness of A are Π_2 properties over $(H(\aleph_1), \in, A)$:

If $A \subseteq P$ in $V[G]$ is universally meager (or null), then there is a $\beta < \omega_2^V$ such that $A \cap V[G_\beta]$ is universally meager (or null) in $V[G_\beta]$.

Moreover, β can be chosen so that, for each Borel set B in $V[G_\beta]$, if $A \cap B \cap V[G] \neq \emptyset$, then $A \cap B \cap V[G_\beta] \neq \emptyset$.

It suffices to see that (in either case), $A \subseteq V[G_\beta]$.

We will show that $A \cap V[G_{\beta+1}] \subseteq V[G_\beta]$. The argument for $A \cap V[G]$ is similar.

Names (I)

Cohen-names and random-names for elements of a Polish space X are given by Borel functions $f\colon 2^\omega \to X$; the realization of the name is $f(g)$, where g is the generic real.

Suppose that $1 \Vdash f(g) \notin V$. Then the f-preimage of each point is meager/null.

If $E \subseteq X$ is universally meager, then $f^{-1}[E]$ is meager.

If $E \subseteq X$ is universally null, then $f^{-1}[E]$ is Lebesgue null.

Names (II)

It follows that:

- If E is universally meager then for any nonmeager Borel B there is a nonmeager Borel $B' \subseteq B$ such that
$$f[B'] \cap E = \emptyset.$$
- If E is universally null then for any non-null Borel B there is a non-null Borel $B' \subseteq B$ such that
$$f[B'] \cap E = \emptyset.$$

Analytic sets and Borel sets

In either case:

$f[B']$ is analytic, so there exist Borel sets F_α $(\alpha < \omega_1)$ such that

$$f[B'] = \bigcup_{\alpha<\omega_1} F_\alpha$$

in any ω_1-preserving outer model.

B' then forces that $g \in B'$, so $f(g)$ will be in some F_α (all of which are disjoint from E).

Proof (IV)

We have then that (in either case) every element of X in

$$V[G_{\beta+1}] \setminus V[G_\beta]$$

is an element of a Borel set in $V[G_\beta]$ disjoint from

$$A \cap V[G_\beta],$$

and therefore also disjoint from A.

A version of the same argument works to show that $A \subseteq V[G_\beta]$.

Note that neither argument proceeded by taking sets which were universally null and not universally meager (or the reverse) and forcing to make the set not univesally null (meager).

I don't know if it is possible to do this (ever or in general).

Universally measurable and categorical sets

We can rerun the same analysis with universally measurable and universally categorical sets.

Since universal measurability and categoricity of A are Π_2 properties over $(H(\aleph_1), \in, A)$:

If $A \subseteq P$ in $V[G]$ is universally categorical (or measurable), then there is a $\beta < \omega_2^V$ such that $A \cap V[G_\beta]$ is universally categorical (or measurable) in $V[G_\beta]$.

Moreover, β can be chosen so that, for each Borel set B in $V[G_\beta]$, if $A \cap B \cap V[G] \neq \emptyset$, then $A \cap B \cap V[G_\beta] \neq \emptyset$.

Names (III)

Fix now a condition B and a Borel function $f\colon B \to X$ representing a name for an element of X.

If $E \subseteq X$ is universally categorical, then $f^{-1}[E]$ is has the Baire Property, so there is a nonmeager Borel $B' \subseteq B$ such that $f[B']$ is either contained in or disjoint from E.

If $E \subseteq X$ is universally measurable, then $f^{-1}[E]$ is Lebesgue measurable, so there is a non-null Borel $B' \subseteq B$ such that $f[B']$ is either contained in or disjoint from E.

In either case, B' forces that $f(g)$ will be in a ground model Borel set which is either contained in or disjoint from E.

Borel reinterpretations (I)

Definition. Given a subset A of a Polish space X, a partial order $\mathbb{P}$ and a V-generic filter G, the *Borel reinterpretation* of A in $V[G]$ is the union of all the (reinterpreted) ground model Borel sets contained in A.

Proposition. A subset A of a Polish space X is universally measurable if and only if, whenever $V[G]$ is an extension of V via random forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

Proposition. A subset A of a Polish space X is universally categorical if and only if, whenever $V[G]$ is an extension of V via Cohen forcing, the Borel reinterpretations of A and $X \setminus A$ in $V[G]$ are complements.

(Now prove Recław's Theorem.)

$\mathbb{G}_0$ graphs again

It follows that iterated random forcing (over a model of GCH) preserves the fact that the universally measurable coloring number of $\mathbb{G}_0$-graphs is at most 3 (and also that there are universally measurable sets without the Baire property).

We shall see that the same holds for Sacks forcing.

What are the universally measurable coloring numbers of $\mathbb{G}_0$-graphs in the Laver model?

(presumably there is a nonmeasurable universally categorical set in the Cohen model)

Cohen and random (III)

Theorem. (Larson-Neeman-Shelah) In an extension of a model of GCH by a length-ω_2 finite-support iteration of random forcing, there are only continuum many universally measurable sets, and a set A is universally measurable if and only if A and its complement are unions of $\aleph_1$-many Borel sets.

Theorem. In an extension of a model of GCH by a length-ω_2 finite-support iteration of Cohen forcing, there are only continuum many universally categorical sets, and a set A is universally measurable if and only if A and its complement are unions of $\aleph_1$-many Borel sets.

Borel reinterpretations (II)

Theorem. The Borel reinterpretation of a universally mesaurable set in an extension by (iterated) random forcing is universally measurable.

To see this (for a single random real), fix a universally measurable $A \subseteq X$ (for some Polish space X) a Borel

$$f \colon 2^\omega \to \mathrm{Meas}(X)$$

(representing a Borel measure in the forcing extension) and a Borel set $D \subseteq 2^\omega$ of positive measure (a condition in random forcing).

Define the Borel measure μ on 2^ω by

$$\mu(E) = \int f(x)(E)\, d\lambda,$$

where λ is Lebesgue measure on 2^ω.

Borel reinterpretations (III)

Since A is universally measurable, there exist Borel B and N such that $A \triangle B \subseteq N$ and $\mu(N) = 0$.

Then $\{x : f(x)(N) > 0\}$ has measure 0, and subtracting it from D we get a condition D' forcing that $f(g)(N) = 0$.

Letting $\hat{A}$ be the Borel reinterpretation of A in the forcing extension, it suffices to see that $\hat{A} \triangle B \subseteq N$ (where B and N are reinterpreted).

This in turn follows from the fact that if E is a Borel set contained in A, then $E \setminus B \subseteq A \setminus B \subseteq N$, and if E is disjoint from A, then $B \cap E \subseteq B \setminus A \subseteq N$.

Sacks forcing

Given a Polish space X, we let $\mathrm{Perf}(X)$ denote the set of perfect subsets of X.

Sacks forcing is $(\mathrm{Perf}(2^\omega), \subseteq)$ (and also the partial order of uncountable Borel sets modulo the ideal of countable sets).

The definitions on the next few slides are due to Ciesielski and Pawlikowski.

$\mathsf{CPA}_{\mathrm{Cube}}(P)$

A *cube* is a continuous injection from $\prod_{n\in\omega} C_n$ to X, where each C_n is a perfect subset of 2^ω.

Let $\mathrm{Cube}(X)$ denote the set of cubes with range contained in X.

Definition. A set $\mathcal{E} \subseteq \mathrm{Perf}(X)$ is Cube-*dense* if for each $f \in \mathrm{Cube}(X)$ there is a $g \in \mathrm{Cube}(X)$ such that $g \subseteq f$ and $\mathrm{range}(g) \in \mathcal{E}$.

The axiom $\mathsf{CPA}_{\mathrm{Cube}}(X)$ says that for every $\mathrm{Cube}(X)$-dense $\mathcal{E} \subseteq \mathrm{Perf}(X)$ there is a $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $|\mathcal{E}_0| \leq \aleph_1$ and

$$|X \setminus \bigcup \mathcal{E}_0| \leq \aleph_1.$$

Ciesielski and Pawlikowski proved that $\mathrm{CPA}_{\mathrm{Cube}}(X)$ holds after a length-ω_2 iteration of Sacks forcing with countable support over a model of CH.

They also showed that it implies that universally null sets have cardinality less than or equal to $\aleph_1$.

Essentially the same proof shows the following.

Theorem. If X is a Polish space and $\mathrm{CPA}_{\mathrm{Cube}}(X)$ holds, then every universally measurable set is the union of at most $\aleph_1$ many sets, each of which is either a perfect set or a singleton.

(no equivalence; what about universally categorical sets?)

Two lemmas

Lemma 1. A set $\mathcal{E} \subseteq \mathrm{Perf}(X)$ is $\mathrm{Cube}(X)$-dense if and only if for every continuous injection

$$f \colon \prod_{n \in \omega} 2^\omega \to P$$

there is a cube $g \subseteq f$ such that $\mathrm{range}(g) \in \mathcal{E}$.

Lemma 2. If D is a Borel subset of $\prod_{n\in\omega} 2^\omega$ of positive Lebesgue measure, then D contains a set of the form

$$\prod_{n \in \omega} C_n,$$

where each $C_n \in \mathrm{Perf}(2^\omega)$.

Proof of the theorem (I)

Let $A \subseteq X$ be universally measurable, and let $\mathcal{E}$ be the collection of perfect subsets of X which are either contained in or disjoint from A. It suffices to see that $\mathcal{E}$ is Cube-dense.

Let

$$f \colon \prod_{n \in \omega} 2^{\omega} \to X$$

be a continuous injection, and let μ be the Borel measure on X defined by letting $\mu(I)$ be the Lebesgue measure of $f^{-1}[I]$.

Proof of the theorem (II)

Since A is universally measurable, there exist Borel subsets B, N of X such that

$$A \triangle B \subseteq N$$

and $\mu(N) = 0$.

Then one of $B \setminus N$ and $(X \setminus B) \setminus N$ has positive μ-measure, so by Lemma 2 there is a function $g \subseteq f$ whose range is contained in either $B \setminus N$ (so contained in A) or $(X \setminus B) \setminus N$ (so contained in $X \setminus A$). $\square$

Tree reinterpretations

A *tree* on X is a set of finite sequences from X closed under initial segments. We write $[T]$ for the set of infinite branches through a tree T.

Given a tree T on $X \times Y$, the *projection* of T ($\mathrm{p}[T]$) is the set of $x \in X^\omega$ for which there is a $y \in Y^\omega$ with $(x, y) \in [T]$.

Given a subset A of a set of the form X^ω, the *tree-reinterpretation* of A in a generic extension is the union of all sets of the form $\mathrm{p}[T]$, where T is a tree in the ground model with $\mathrm{p}[T] \subseteq A$ in the ground model.

This is the same as the Borel reinterpretation in the case where the extension is via a c.c.c. forcing.

universally Baire sets

Definition. Given a partial order $\mathbb{P}$ and a set X, a set $A \subseteq X^\omega$ is $\mathbb{P}$-*Baire* if, in any forcing extension by $\mathbb{P}$, the tree-reinterpretations of A and $X^\omega \setminus A$ are complements.

Using this definition, universal measurability is random-Baireness and universal categoricity is Cohen-Baireness.

A set is said to be *universally Baire* if it is $\mathbb{P}$-Baire for all partial orders $\mathbb{P}$.

Questions

(Kumar) Does ZFC prove that there is a universally measurable set which is not universally Baire? (ZFC + a Woodin cardinal does)

Is it consistent with ZFC that every universally measurable set has universally null symmetric difference with some universally Baire set?

Is it consistent with ZFC that the universally measurable sets are the members of the smallest σ-algebra containing the universally Baire sets and the universally null sets (and closed under the Suslin operation)?

Is it consistent with ZFC that there exists a universally Baire set of cardinality $\aleph_2$ which does not contain a perfect set (yes for $\aleph_1$)?

Polar ideals

Definition. Let I be a σ-ideal on a Polish space X which is generated by analytic sets. We say that I is *polar* if there is a set $M \subseteq \mathrm{Meas}(X)$ such that

$$I = \{A \subseteq X : \forall \mu \in M\, \mu(A) = 0\}.$$

We say that I is $\mathbf{\Sigma}^1_1$-*polar* if there exists an analytic $M \subseteq \mathrm{Meas}(X)$ witnessing that I is polar.

If the quotient forcing $P_I = \mathrm{Borel}(X)/I$ is proper in all forcing extensions, then we say that I is an *iterable* $\mathbf{\Sigma}^1_1$-polar ideal, and we call P_I a *polar forcing*.

Examples of polar forcings

Random forcing (the Lebesgue-null ideal)

Sacks forcing (the ideal of countable sets, induced by letting $M = \mathrm{Meas}(X)$)

The countable product of Sacks forcing. This is equivalent to $\mathrm{Borel}((2^\omega)^\omega)/I$, where I is the ideal of subsets of $(2^\omega)^\omega$ not containing a product of perfect sets. M is the set of product measures concentrating on a perfect set in each coordinate (Lemma 2 above shows that this works).

The Borel sets modulo the ideal generated by the analytic $\mathbb{E}_0$-selectors.

Iterations

Theorem (Larson-Zapletal) A countable iteration of polar forcings is polar.

The successor step

Given $B \subseteq X_0 \times X_1$, and $x \in X_0$, $B_x = \{y : (x, y) \in B)\}$.

Definition. Suppose that, for each $i \in \{0, 1\}$, $M_i \subseteq \mathrm{Meas}(X_i)$, for some Polish space X_i. We define $M_0 * M_1$ to be the set of measures on $X_0 \times X_1$ of the form

$$\nu(B) = \int f(x)(B_x)\, d\mu,$$

where $\mu \in M_0$ and $f \colon X_0 \to M_1$ is Borel.

Lemma. Suppose that, for each $i \in 0, 1$, I_i is an iterable $\mathbf{\Sigma}^1_1$-polar ideal on a Polish space X_i as witnessed by $M_i \subseteq \mathrm{Meas}(X_i)$. Then $M_0 * M_1$ an analytic set of measures giving rise to an iterable $\mathbf{\Sigma}^1_1$-polar ideal $I_0 * I_1$ on $X_0 \times X_1$. Moreover, $P_{I_0} * P_{I_1}$ is forcing-equivalent to $P_{I_0 * I_1}$.

The limit step

The $*$ operation on sets of measures is associative.

Given $M_i \subseteq \mathrm{Meas}(X_i)$ $(i \in \omega)$ we let $*_i M_i$ be the set of $\mu \in \mathrm{Meas}(\prod_{i\in\omega} X_i)$ such that, for each $j \in \omega$, the preimage measure induced by the projection of $\prod_{i\in\omega} X_i$ to $\prod_{i\leq j} X_i$ and μ is in $*_{i\leq j} M_i$.

Then $*_{i\in\omega} M_i$ an analytic set of measures giving rise to an iterable $\mathbf{\Sigma}^1_1$-polar ideal $*_{i\in\omega} I_i$ on $X_0 \times X_1$. Moreover, the full-support iteration of the partial orders P_{I_i} is forcing-equivalent to $P_{*_{i\in\omega} I_i}$.

Measured extensions (I)

Definition. Let $M \subseteq N$ be transitive models of set theory with the same ordinals. We say that N is a *measured extension* of M if the following hold.

- Every $a \subseteq M$ which is a countable set in N is a subset of a $b \in M$ which is countable in M.
- Suppose that $\epsilon \in \mathbb{Q}^+$, X is a Polish space in M, and F is, in M, a collection of Borel subsets of X. If there is a Borel probability measure μ on X in the model N such that for every $B \in F$, $\mu(B) \geq \epsilon$, then there is a Borel probability measure ν in the model M such that for every $B \in F$, $\nu(B) \geq \epsilon$.

Measured extensions (I)

Measured extensions are ω^ω-bounding and preserve outer measure, and the relation of being a measured extension is transitive.

If N is a measured extension of M, and A is, in M, a universally measurable subset of a Polish space X, then the Borel reinterpretations of A and $X \setminus A$ are complements in N.

Moreover, then Borel reinterpretation of A is universally measurable in N.

Theorem. (Larson-Zapletal) Let I be an iterable Σ^1_1-polar ideal on a Polish space X. Any forcing extension by an countable support iteration of the partial order P_I is measured.

Proof sketch

We sketch the proof for a single forcing P_I, which gives the theorem for countable iterations by the iterability of polar ideals. The general case follows from standard forcing arguments.

Suppose that

- Y is a Polish space,
- F is a set of Borel subsets of Y,
- $\epsilon \in \mathbb{Q}^+$,
- $p \in P_I$ is a condition and
- τ is a P_I-name for a measure on Y such that, for all $B \in F$, $p \Vdash \tau(B) \geq \epsilon$.

We may also fix a Borel function $f : p \to \mathrm{Meas}(Y)$ such that $p \Vdash \tau = f(g)$.

Fix a probability measure ν on X such that $\nu(p) = 1$ and all analytic sets in I have ν-measure 0.

Let μ be the Borel probability measure on Y defined by

$$\mu(B) = \int f(x)(B)d\nu(x).$$

Suppose towards a contradiction that $\mu(B) < \epsilon$ for some $B \in F$. Then

$$r = \{x \in p : f(x)(B) < \epsilon\}$$

has positive ν-measure and r forces that

$$f(g)(B) = \tau(B) < \epsilon,$$

giving a contradiction. $\square$

Projective sets

Recall that a set is $\mathbf{\Sigma}^1_1$ it is a continuous image of a Borel set, and $\mathbf{\Pi}^1_1$ if it is the complement of a $\mathbf{\Sigma}^1_1$ set.

More generally, a set is

- $\mathbf{\Sigma}^1_{n+1}$ if it is a continuous image of a $\mathbf{\Pi}^1_n$ set,
- $\mathbf{\Pi}^1_{n+1}$ if it is the complement of a $\mathbf{\Sigma}^1_{n+1}$ set and
- $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Each class $\mathbf{\Sigma}^1_{n+1} \setminus \mathbf{\Delta}^1_n$ is nonempty.

The *projective sets* are the members of $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_1$.

The influence of large cardinals

If there exist infinitely many Woodin cardinals, then every projective set is universally measurable and universally categorical (and moreover, $\mathbb{P}$-Baire for all $\mathbb{P}$ of cardinality less than the supremum of the Woodin cardinals).

If there exists one Woodin cardinal, then no universally null set of cardinality $\aleph_1$ is universally Baire.

Theorem. (Larson-Shelah) If there exists an $r \subseteq \omega$ such that $V = L[r]$, then there is a measured proper forcing extension in which evey universally measurable set is $\mathbf{\Delta}^1_2$ and every universally categorical set is $\mathbf{\Delta}^1_2$.

Proof strategy

The idea of the proof is : iterate to make

$$[\omega^\omega]^{\aleph_1} \subseteq \mathbf{\Delta}^1_2$$

and everthing will work itself out.

The hypothesis "$V = L[r]$ for some $r \subseteq \omega$" is used to produce an absolutely $\Delta^1_2(r)$ set of reals coding a ladder system

$$\bar{C} = \langle C_\alpha : \alpha < \omega_1 \rangle$$

on ω_1 (a choice of a cofinal subset of ordertype ω for each countable limit ordinal).

Pathological functions

Given an ideal I on 2^ω, we say that a partial function

$$F\colon 2^\omega \to 2$$

is *I-pathological* if, whenever

- $s \in 2^{<\omega}$,
- $i < 2$ and
- $B \subseteq [s]$ is Borel and I-positive,

there is a $x \in B \cap \mathrm{dom}(F)$ such that $s \lhd x$ and $F(x) = i$.

We say *null-pathological* when I is the ideal of Lebesgue-null sets, and *totally pathological* when I is the ideal of countable sets.

ZFC implies the existence of totally pathological functions, and if $r \subseteq \omega$ is such that $\mathbf{V} = \mathbf{L}[a]$, then there exists such a function F which is $\Delta^1_2(r)$.

The iteration

The forcing construction is a countable support iteration of partial orders $Q_{\bar{C},F,g}$ which produce (by countable initial segments), given

$$g \colon \omega_1 \to 2,$$

a function

$$h \colon \omega_1 \to 2$$

such that, for all countable limit ordinals α,

$$F(h{\restriction}C_\alpha) = g(\alpha).$$

The partial orders $Q_{\bar{C},F,g}$ are proper and do not add reals.

However, for each $X \in [\omega^\omega]^{\aleph_1}$ there is a length-ω iteration of partial orders of the form $Q_{\bar{C},F,g}$ producing a real coding X relative to $\bar{C}$ and F, and thereby making X $\mathbf{\Delta}^1_2$. .

Everything works itself out

As in the Larson-Neeman-Shelah proof, one uses the fact that, for every universally measurable set X in the forcing extension $\mathbf{V}[G]$ (after the iteration), there is an intermediate extension $\mathbf{V}[G_\alpha]$ in which $X \cap \mathbf{V}[G_\alpha]$ is universally measurable.

That is, for every Borel measure (or continuous function), there is a Borel set which has been forced to witness the corresponding instance of universal measurability for (the eventual) X.

These Borel sets determine X. As in the case of random forcing this is shown by using the fact that for reals induce measures.

Local randomness

In the current case, one uses the fact that for any condition p in our iteration P and any countable elementary submodel M of a large enough set (with $P \in M$) there exists a finitely branching tree T of height ω consisting of conditions in $P \cap M$ below p such that for any Borel set $E \subseteq [T]$ of positive Lebesgue measure, some $q \leq p$ forces that the generic filter will contain an M-generic element of E.

Each real in $\mathbf{V}[G] \setminus \mathbf{V}[G_\alpha]$ is then forced into a suitable Borel set in $\mathbf{V}[G_\alpha]$ by applying the universal measurability of $X \cap \mathbf{V}[G_\alpha]$ to a measure induced by Lebesgue measure on T and a name for the real.

Preserving pathology

To make this argument work, we need that the function F remains null-pathological throughout the iteration. This follows from the preservation of Lebesgue outer measure under countable support iterations of proper forcings not adding reals.

Generalizing to other σ-algebras

Given a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\omega^\omega)$, say that a set $X \subseteq \omega^\omega$ is *universally*-$\mathcal{A}$ if every continuous preimage of X in ω^ω is in $\mathcal{A}$.

Given an ideal I on ω^ω, let $\mathcal{A}_I$ be the σ-ideal of subsets X of ω^ω such that $X \triangle B \in I$ for some Borel set B.

The argument just given applies to $\mathcal{A}_I$ for any Borel ideal I such that

- countable support iterations of partial orders (individually) not adding reals preserve I-positivity and
- the analogous form of Fubini's theorem holds.

This includes the meager ideal.

In the resulting model, every universally measurable set is a union of $\aleph_1$ many Borel sets (in addition to being $\boldsymbol{\Delta}^1_2$) and so is universally categorical set.

However, there are universally null sets which are not meager, and universally meager sets which are not Lebesgue null.

The forcing extension is measured (so universally measurable sets reinterpret as universally measurable sets), although it is not evidently induced by polar forcings.

References

K. Ciesielski, L. Pawlikowski, **The Covering Property Axiom, CPA**, Cambridge University Press, 2004

P. Larson, I. Neeman, S. Shelah, *Universally measurable sets in generic extensions*, Fundamenta Mathematicae 208 (2010) 2, 173-192

P. Larson, S. Shelah, *Universally measurable sets may all be* $\boldsymbol{\Delta}^1_2$, Fundamenta Mathematicae 266 (2024), 97-120

P. Larson, J. Zapletal, *Polar forcings and measured extensions*, Topology and Its Applications 323, January 2023, article 108290

B. Miller, *Measurable chromatic numbers*, J. Symbolic Logic 73 (2008), no. 4, 1139–1157