Universally Measurable Sets I

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Measures

Given a set X, a *σ-algebra* on X is a nonempty collection of subsets of X which is closed under countable unions and complements.

A *measure* on X is a function μ from some σ-algebra on X to $[0, \infty]$, such that $\mu(\emptyset) = 0$ and, whenever A_n $(n \in \omega)$ are disjoint members of the domain of μ,

$$\mu(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} \mu(A_n).$$

Types of measures

A measure μ on a set X is

- *finite* if $\mu(X)$ is finite;
- σ-*finite* if X is a countable union of sets with finite measure;
- a *probability measure* if $\mu(X) = 1$;
- *atomless* if $\mu(a) = 0$ for all finite $a \subseteq X$.

Borel measures

Given a topological space (X, τ), the Borel subsets of X are the members of the smallest σ-algebra on X containing the open sets.

We will write $\mathrm{Borel}(X)$ for the set of Borel subsets of X.

A *Borel measure* on (X, τ) is a measure whose domain contains the Borel sets.

Complete measures

A measure μ on a set X is *complete* if $A \in \mathrm{dom}(\mu)$ whenever there exist $B, N \in \mathrm{dom}(\mu)$ such that $A \triangle B \subseteq N$ and $\mu(N) = 0$ (in which case $\mu(A) = \mu(B)$).

The *completion* of a measure is the smallest complete measure containing it (i.e., the intersection of all the complete measures containing it).

Two measures

- Lebesgue measure on $\mathbb{R}$ (σ-finte and atomless; its restriction to $[0,1]$ is a probability measure; (a,b) gets measure $b-a$);
- Haar measure on 2^ω (an atomless probability measure; $[s]$ gets measure $2^{-|s|}$).

Borel functions

A function between topological spaces is *Borel* if the preimage of each open set is Borel.

If μ is a Borel measure on a topological space X, and

$$f : X \to \mathbb{R}^{\geq 0}$$

is a Borel function, then

$$\nu(A) = \int_A f(x)\, d\mu$$

defines another Borel measure on X.

Analytic sets

Definition. A subset of $\mathbb{R}$ (or, more generally, of any Polish space) is *analytic* (or $\mathbf{\Sigma}^1_1$) if it is a continuous image of a Borel set, and *coanalytic* (or $\mathbf{\Pi}^1_1$) if it is the complement of an analytic set.

By a classical theorem of Lusin, analytic subsets of $\mathbb{R}$ or 2^ω are Lebesgue/Haar measurable (so coanalytic sets are too).

Nonmeasurable sets

Four clasical ways in which nonmeasurable sets arise (using the Axiom of Choice):

- Choosing a member from each equivalence class of $\mathbb{R}/\mathbb{Q}$ (forming a Vitali set).
- Fubini's Theorem, which can be used to show that no wellordering of $\mathbb{R}$ is Lebesgue measurable.
- Nonprincipal ultrafilters on ω.
- Rapid filters.

Universally measurable sets

Definition. A subset of a topological space is *universally measurable* if it is measurable with respect to (i.e., in the domain of) every complete σ-finite Borel measure on the space.

We will write $\mathrm{UMeas}(X)$ for the set of universally measurable subsets of a topological space (X, τ).

For any topological space (X, τ), $\mathrm{UMeas}(X)$ is a σ-algebra containing the Borel sets.

The set of universally measurable subsets of a topological space would not change if one replaced "σ-finite" with "finite" or "probability".

It would also not change if one added the requirement that the measures be atomless (since a σ-finite measure can have only countably many atoms).*

Polish spaces

A topological space (X, τ) is *completely metrizable* if there exists a metric d on X inducing τ relative to which X is complete (i.e., such that every d-Cauchy sequence converges).

A *Polish space* is a completely metrizable separable topological space.

The class of Polish spaces is closed under countable products and taking G_δ subsets.

Examples

Examples of Polish spaces:

- $\mathbb{Z}$
- $\mathbb{R}^n$
- $[0,1]$
- 2^ω
- $(2^\omega)^\omega$
- ω^ω

The Suslin operation

The *Suslin operation* applied to sets A_σ ($\sigma \in \omega^{<\omega}$) results in the set

$$\bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} A_{x \restriction x}.$$

The *C-sets* are the members of the smallest σ-algebra containing the open sets and closed under the Suslin operation.

If (X, τ) is Polish then $\mathrm{UMeas}(X)$ is closed under the Suslin operation and therefore contains the C-sets.

The questions we consider in these lectures generally have the following form: what can the set of universally measurable subsets of Polish spaces be like in models of ZFC?

Universally measurable homomorphisms

Given topological spaces X and Y, a function $f\colon X \to Y$ is *universally measurable* if $f^{-1}[A]$ is universally measurable for every Borel $A \subseteq Y$.

Theorem (Rosendal) If π is a universally measurable homomorphism from a Polish group to a separable group then π is continuous.

Theorem (Christensen) If $G = \bigcup_{i\in\mathbb{N}} A_i$ is a covering of a Polish group by universally measurable sets, then for every neighborhood U of the identity there exist $k \in \mathbb{N}$ and $g_1, \ldots, g_n \in U$ such that

$$\bigcup_{i=1}^{n} g_i A_k A_k^{-1} g_i^{-1}$$

is a neighborhood of the identity.

The Isomorphism Theorem (for measures)

Theorem. If X and Y are Polish spaces and μ and ν are atomless Borel probability measures on X and Y respectively, then there is a Borel bijection

$$f : X \to Y$$

such that

$$\mu(I) = \nu(f[I])$$

for all Borel $I \subseteq X$.

The Isomorphism Theorem gives us the following:

- For the sorts of questions that we will be asking, it doesn't matter which Polish space we consider (often we use 2^ω).
- Since analytic subsets of $[0, 1]$ are Lebesgue measurable, and a Borel preimage of an analytic set is analytic, analytic subsets of any Polish space are universally measurable.

Preimage measures

If

- μ is a Borel measure on a topological space (X, τ),
- (Y, σ) is a topological space,
- $f : X \to Y$ is a Borel function,

then $\mu_f(A) = \mu^{-1}[A]$ defines a Borel measure on (Y, σ).

This is sometimes called the *pushforward* measure induced by f.

It follows that

- a subset A of Polish space X is universally measurable if and only if $f^{-1}[A]$ is Lebesgue measurable whenever $f\colon 2^\omega \to X$ is a Borel function.
- If (X, τ) and (Y, σ) are Polish spaces, $A \subseteq Y$ is universally measurable and $f\colon X \to Y$ is Borel, then $f^{-1}[A]$ is universally measurable.
- In a model of ZF in which every set of reals is Lebesgue measurable (such as a Solovay model), every set of reals is universally measurable.

Universally Null Sets (I)

A subset of a topological space is *universally null* if is has measure 0 with respect to every atomless σ-finite Borel measure on the space.

$\mathrm{UNull}(X)$ denotes the set of universally null subsets of (X, τ).

Again, the collection of universally null sets does not change if we require the measure to be finite, or a probability measure.

Universally Null Sets (II)

$\mathrm{UNull}(X)$ is a σ-ideal on X (i.e., it is closed under subsets and countable unions) containing all countable sets.

$\mathrm{UNull}(X) \subseteq \mathrm{UMeas}(X)$

A subset A of Polish space X is universally null if and only if $f^{-1}[A]$ is Lebesgue null whenever

$$f : 2^{\omega} \to X$$

is a Borel function such that the preimage of each point is Lebesgue null.

Perfect sets (I)

A subset of a topological space is said to be *perfect* if it is closed, uncountable, and contains no isolated points.

We will write $\mathrm{Perf}(X)$ for the set of perfect subsets of (X, τ).

When (X, τ) is a Polish space, each element of $\mathrm{Perf}(X)$ has cardinality $2^{\aleph_0}$.

For each $P \in \mathrm{Perf}(X)$, there is a Borel probabillity measure μ on X with $\mu(P) = 1$.

It follows that a universally null subset of a Polish space cannot contain a perfect set.

Perfect sets (II)

Every uncountable analytic subset of a Polish space contains a perfect set (so the analytic universally null sets are exactly the countable sets).

Theorem (Grzegorek, Ryll-Nardzewski) The (coanalytic) set of wellorderings of ω is universally measurable but does not have universally null symmetric difference with any Borel set.

Category

A subset of a topological space is

- *nowhere dense* if its closure does not contain any nonempty open set, and
- *meager* if it is a countable union of nowhere dense sets.

A set has the Baire Property if its symmetric difference with some open set is meager.

Universally categorical sets

By analogy with the measure case, we make the following definitions.

A subset A of a Polish space (X, τ) is *universally categorical* if $f^{-1}[A]$ has the Baire property whenever (Y, σ) is a Polish space and $f\colon Y \to X$ is Borel.

A subset of a Polish space is *universally meager* if $f^{-1}[A]$ is meager whenever (Y, σ) is a Polish space and $f\colon Y \to X$ is Borel such that the preimage of each point is meager.

We write $\mathrm{UCat}(X)$ for the set of universally categorical subsets of X, and $\mathrm{UMeag}(X)$ for the set of universally meager sets.

When (X, τ) is a Polish space, $\mathrm{UCat}(X)$ is a σ-algebra (moreover, a Suslin algebra) containing the analytic sets, and $\mathrm{UMeag}(X)$ is a σ-ideal containing the countable sets.

Recław's Theorem

Theorem.(Recław) Let (X, τ) be a Polish space. A subset of X which is wellordered by a universally measurable relation on X is universally null. A subset of X which is welllordered by a a universally categorical relation on X is universally meager.

The proof uses Fubini's theorem and (its category-analogue) the Kuratowski-Ulam theorem.

Uncountable universally null and meager sets (I)

The relation of mod-finite domination is Borel (so universally measurable and universally categorical) and contains chains of ordertype $\mathfrak{b}$.

It follows (from Recław's theorem) that

$$\mathrm{UNull}(\omega^\omega) \cap \mathrm{UMeag}(\omega^\omega)$$

contains sets of cardinality $\mathfrak{b}$, and (since both properties are closed under taking subsets) at least $2^{\mathfrak{b}}$ many such sets.

Uncountable universally null and meager sets (II)

Similarly, one show that for any ideal $\mathcal{I}$ on any Polish space P, if $\mathcal{I}$ is generated by sets of some fixed Borel complexity, then there is a set of cardinality

$$\min\{\mathsf{non}(\mathcal{I}), \mathsf{cov}(\mathcal{I})\}$$

which is wellordered by a Borel relation (and is therefore both universally null and universally meager).

Cihoń's diagram

$$
\begin{array}{ccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\aleph_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}
$$

Additivity

For any Polish space (X, τ), $\mathrm{UMeas}(X)$ and $\mathrm{UNull}(X)$ are both closed under unions of cardinality less than $\mathrm{Add}(\mathcal{N})$.

Similarly, $\mathrm{UCat}(X)$ and $\mathrm{UMeag}(X)$ are both closed under unions of cardinality less than $\mathrm{Add}(\mathcal{M})$.

Grzegorek's Theorem

Theorem. (Grzegorek) There exist a universally null set of cardinality non($\mathcal{N}$) and a universally meager set of cardinality non($\mathcal{M}$).

One can use this theorem to show that the classes of universally measurable and universally categorical (check) sets are not closed under projections.

Questions

Is it consistent with ZFC that every universally measurable set has the property of Baire (in which case it would be universally categorical)?

Is it consistent with ZFC that every universally categorical set is universally measurable?

More Questions

If is consistent with ZFC that $\mathrm{UMeas}(X)$ is the same as the C-sets? (no way)

Is it consistent with ZFC that every universally measurable set has universally null symmetric difference with some C-set?

Is it consistent with ZFC that the universally measurable sets are the members of the smallest σ-algebra closed under the Suslin operation and containing the open sets and the universally null sets?

Another Question

Is it consistent with ZFC that $\mathrm{UNull}(X) = \mathrm{UMeag}(X)$ (for some/any Polish space (X, τ))?

Recall that the real line is the union of a meager set and a null set, so there are nonmeager null sets and vice versa.

Nonmeager universally null sets

Since universally null sets don't contain perfect sets, a universally null set which has the Baire property must be meager.

Theorem.(Brendle-Larson) If $\min\{\mathfrak{d}, \text{non}(\mathcal{N})\} = \text{cof}(\mathcal{M})$, then there is a nonmeager universally null set of cardinality $\text{cof}(\mathcal{M})$.

We build a nonmeager universally null set in the Polish space ω^ω. A key point is that if μ is any Borel probability measure on ω^ω, then there is a $g \in \omega^\omega$ such that $\mu(B_g) = 1$, where

$$B_g = \{f : f \leq^* g\}.$$

Superperfect trees

A tree $T \subseteq \omega^{<\omega}$ is *superperfect* if, for densely many $s \in T$,

$$\{n : s^\frown\langle n\rangle \in T\}$$

is infinite.

We write $[T]$ for the set of infinite paths through T.

If $A \subseteq \omega^\omega$ is meager, then there is a superperfect tree

$$T \subseteq \omega^{<\omega}$$

such that $A \cap [T] = \emptyset$.

Proof of the theorem (I)

Let κ be $\mathrm{cof}(\mathcal{M})$, which is equal to $\mathfrak{d}$ by the hypothesis.

Let $\{A_\alpha : \alpha < \kappa\}$ witness $\mathrm{cof}(\mathcal{M}) = \kappa$ and let $\{f_\alpha : \alpha < \kappa\}$ be a dominating family in ω^ω.

For each $\alpha < \kappa$, let $T_\alpha \subseteq \omega^{<\omega}$ be a superperfect tree such that

$$A_\alpha \cap [T_\alpha] = \emptyset.$$

Claim. Since $\mathfrak{d} = \kappa$, we can pick for each $\alpha < \kappa$ an $x_\alpha \in [T_\alpha]$ such that, for all $\beta < \alpha$, $f_\beta \not\geq^* x_\alpha$.

Proof of the theorem (II)

Since $X = \{x_\alpha : \alpha < \kappa\}$ is not contained in any A_α, it is nonmeager.

To see that X is universally null, let μ be a Borel probability measure on ω^ω. Fix $g \in \omega^\omega$ such that $\mu(B_g) = 1$.

Let α be such that $g \leq^* f_\alpha$. Then

$$\{x_\beta : \beta > \alpha\} \cap B_g = \emptyset.$$

Thus $\mu(\{x_\beta : \beta > \alpha\}) = 0$. Since $\mathrm{non}(\mathcal{N}) \geq \kappa$,

$$\mu(\{x_\beta : \beta \leq \alpha\}) = 0.$$

It follows that $\mu(X) = 0$. □

Proof of the claim

Given $f : \omega \to \omega$ and superperfect $T \subseteq \omega^{<\omega}$, let $f_T : \omega \to \omega$ be such that, whenever s is an nth-level splitnode of T, and $s(k) \leq f(k)$ for all $k < |s|$, $f_T(n) > f(|s|)$.

Then, whenever s is an nth-level splitnode of T, there is a $k \leq |s|$ such that $(s^\frown\langle f_T(n)\rangle)(k) > f(k)$.

Given $F = \{f_\alpha : \alpha < \beta\}$ and T, let F' be the set of $f \in \omega^\omega$ which agree with some $f \in F$ mod-finite. Let $h \in \omega^\omega$ be such that, for all $f \in F'$, $f_T \not\geq^* h$.

Let x_α extend infintely many splitnodes of T, in such a way that, for each initial segment s of x_α, if s is an nth level splitnode of T, then $x_\alpha(n) = h(n)$.

Question

Does the existence of a nonmeager unviersally null set follow from either $\mathfrak{d} = \mathrm{cof}(\mathcal{M})$ or non$(\mathcal{N}) \geq \mathrm{cof}(\mathcal{M})$?

Note that if

$$\text{non}(\mathcal{N}) > \mathrm{cof}(\mathcal{M})$$

then

$$\text{non}(\mathcal{N}) > \text{non}(\mathcal{M}),$$

in which case there are nonmeager universally null sets of cardinality non$(\mathcal{M})$. So the interesting case is when

$$\text{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}).$$

$\mathbb{G}_0$ graphs

For $D \subseteq 2^{<\omega}$, let $G(D)$ be the graph on 2^{ω} connecting x and y if they disagree in exactly one place, and their longest comment initial segment is in D.

A $\mathbb{G}_0$-graph is a graph of the form $G(D)$ when D is dense and has exactly one member of length n for each $n \in \omega$. We let $\mathbb{G}_0$ be the set of $\mathbb{G}_0$-graphs.

The $\mathbb{G}_0$-dichotomy (Solecki-Kechris-Todorceivic) says that if $G \in \mathbb{G}_0$ and H is an analytic graph on a Polish space, then either H has a Borel $\mathbb{N}$-coloring or there is a homomorphism from G to H.

More on $\mathbb{G}_0$-graphs

Each $G \in \mathbb{G}_0$ is connected, acyclic and locally countable.

Comeagerly many $x \in 2^\omega$ lie in a connected component of G whose members all have infinite degree.

Measure 1 many $x \in 2^\omega$ have finite degree in G.

Baire colorings

Given a graph G on a Polish space P and a collection of sets Γ, let $\chi(G,\Gamma)$ be the smallest cardinality of a set of G-independent sets from Γ whose union contains P.

If $G \in \mathbb{G}_0$ and $A \subseteq 2^\omega$ is G-independent, then for no $s \in 2^{<\omega}$ is

$$A \cap [s]$$

comeager in s.

Letting Δ_B be the collection of subsets of 2^ω with the Baire Property, it follows that

$$\chi(G, \Delta_\mathrm{B}) \geq \mathrm{cov}(\mathcal{M}).$$

Measurable colorings

Let Δ_{U} be the set of universally measurable subsets of 2^ω.

Ben Miller has shown that, consistently, for any $G \in \mathbb{G}_0$,

$$\chi(G, \Delta_{\mathrm{U}}) \leq 3.$$

It is an open question whether ZFC implies that $\chi(G, \Delta_{\mathrm{U}}) \leq 3$ for any $G \in \mathbb{G}_0$.

Free functions

A function $f : X \to X$ is said to be *free* if, for all $x \in X$ and $m \in \mathbb{N}$,

$$f^{(m)}(x) \neq x.$$

The graph $\mathcal{G}_f$ connects each $x \in X$ with $f(x)$.

Theorem. $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$ if and only if there exists a universally measurable set $A \subseteq X$ such that for every $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

Given $x \in X$, we call $\{f^i(x) : i \in \omega\}$ the *f-forward image* of x.

The proof of of Miller's theorem involves 3-coloring part of the graph G in a Borel way, and then proving the following lemma, which implies that $\chi(\mathcal{G}_f, \Delta_{\mathrm{U}}) \leq 3$.

Lemma. (B. Miller) Suppose that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$. Let X be a Polish space, $f\colon X \to X$ be a free Borel functon, and

$$\langle F_n : n \in \omega \rangle$$

be a $\subseteq$-increasing sequence of Borel equivalence relations whose union contains $\mathcal{G}_f$, such that no F_n contains any set of the form

$$\{f^i(x) : i \in \omega\}.$$

Then there exists a universally measurable set $A \subseteq X$ such that, for all $x \in X$ there exist $i, j \in \omega$ such that $f^i(x) \in A$ and $f^j(x) \notin A$.

We will prove the lemma and improve the cardinal characteristic assumption used (the improvement is joint with Michel Gaspar).

Given x, let $e(x)$ be the least n such that $xF_nf(x)$. For every x,

$$\{e(f^i(x)) : i \in \omega\}$$

is infinite.

It follows that any Borel probability measure μ, any $n \in \omega$ and any $\epsilon > 0$, there is an $m > n$ such that

$$\mu(\{x : \exists i\, e(f^i(x)) \in (n, m]\}) > 1 - \epsilon.$$

Given μ, let g_μ be such that, for each $n \in \omega$,

$$\mu(\{x : \exists i\, e(f^i(x)) \in (n, g_\mu(n)]\}) > 1 - (1/n).$$

Given $A \subseteq \omega$, let D_A be the set of x in X such that

$$\{i : e(f^i(x)) \in A\}$$

and

$$\{i : e(f^i(x)) \notin A\}$$

are both infinite.

Each D_A is Borel and f-invariant.

Lemma. Let μ be a Borel measure on 2^ω and let $A \subseteq \omega$ be such that A contains infinitely many intervals of the form $(n, g_\mu(n)]$, and is also disjoint from infinitely many such intervals. Then $\mu(D_A) = 1$.

$\max\{\mathfrak{b}, \mathfrak{s}\}$

Let γ be the least cardinality of a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that, for every increasing $f : \omega \to \omega$ there exists an $A \in \mathcal{A}$ such that the sets

$$\{n \in \omega : (n, f(n)] \subseteq A\}$$

and

$$\{n \in \omega : (n, f(n)] \cap A = \emptyset\}$$

are both infinite.

Then

$$\gamma = \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d},$$

where $\mathfrak{s}$, the *splitting number*, is the smallest cardinality of a set $\mathcal{B} \subseteq \mathcal{P}(\omega)$ such that, for all infinite $A \subseteq \omega$ there exists $B \in \mathcal{B}$ with $A \cap B$ and $A \setminus B$ infinite.

Fix a family $\mathcal{A} = \langle A_\alpha : \alpha < \gamma \rangle$ witnessing the value of γ.

Each D_{A_α} is Borel, and thus universally measurable.

If $\gamma \leq \mathrm{add}(\mathrm{null})$, then

$$\bigcup_{\beta<\gamma} (D_{A_\beta} \setminus \bigcup_{\alpha<\beta} D_{A_\alpha})$$

is a universally measurable set as desired.

$\mathfrak{c} = \aleph_2$?

$\mathfrak{s} \leq \text{non}(\mathcal{M}), \mathfrak{d}, \text{non}(\mathcal{L})$.

If $\mathfrak{c} = \aleph_2$ and every universally measurable set has the Baire property, then

- $\aleph_2 = \max\{\mathfrak{b}, \mathfrak{s}\} > \text{add}(\mathcal{N}) = \aleph_1$ (by Miller-Gaspar-Larson) so
- $\aleph_2 = \mathfrak{d} = \text{non}(\mathcal{M}) = \text{cof}(\mathcal{M})$ so
- $\aleph_1 = \text{non}(\mathcal{N}) < \text{cof}(\mathcal{M})$ (by Brendle-Larson) so
- $\mathfrak{s} = \aleph_1$ and $\mathfrak{b} = \aleph_2$.

This collection of values holds in the Laver and random/Laver models.

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