The cohomology of the ordinals
day 3

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Corollary

It is consistent relative to the existence of a strongly compact cardinal that, for every nonzero abelian group A,

$$\mathrm{H}^1(\xi; \mathcal{A}) \neq 0 \text{ if and only if } \mathrm{cf}(\xi) = \omega_1$$

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The conclusions hold in any model of the P-ideal Dichotomy (and, in particular, of PFA) as well. But in all of these models, $\mathrm{H}^2(\omega_2; \mathcal{A}) \neq 0$ for any nonzero abelian group A.

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Theorem (assuming large cardinals)

$\mathrm{Con}(\mathsf{ZFC} + \mathrm{H}^n(\omega_m; \mathcal{A}) = 0$ *whenever* $n \neq m$ *and* $n > 0)$.

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One wonders more generally exactly which configurations of these groups' behaviors are possible — and what we may infer of the ambient universe from any given one.

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Another: *Is an algebraic framing of what's ultimately a story of infinitary combinatorics actually helpful — and if so, how?*

These are genuinely excellent questions.

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*When $A = \bigoplus_{\omega_n} \mathbb{Z}$, this kind of "massage" can give us $\mathbb{Z}$ for codomain, but at the price of some expansion of domain: $\mathrm{H}^n(\omega_n^2; \mathbb{Z}) \neq 0$ *is* a ZFC theorem for all $n \geq 0$.

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Frequently useful is the following convention: a function f from an ordinal ε to $\{0,1\}$ is

- *0-coherent* if $f|_\delta$ is finitely supported for all $\delta < \varepsilon$, and
- *0-trivial* if f itself is finitely supported.

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Similarly, a coherent family of finite-to-one functions $f_\varepsilon : \varepsilon \to \omega$ may be construed, by way of the reflections of the characteristic functions of their graphs (i.e., via the collections of pairs $(f_\varepsilon(\xi), \xi)$ $(\xi < \varepsilon)$), as a family of functions $\varphi_\varepsilon : \omega \times \varepsilon \to \mathbb{Z}/2$ exhibiting

(0.1) *horizontal* 0-*coherence*, meaning that the functions $\varphi_\varepsilon|_{n\times\varepsilon}$ are finitely supported for any $n < \omega$,

(0.2) *vertically persistent horizontal non*-0-*triviality*, meaning that for all $\delta < \omega_1$ there exists an $\varepsilon < \omega_1$ such that $\varphi_\varepsilon\big|_{\omega\times[\delta,\varepsilon)}$ is infinitely supported, and

(0.3) *horizontally* 0-*trivial vertical differences*, meaning that the functions $\varphi_\varepsilon - \varphi_\delta$ are finitely supported for all $\delta < \varepsilon < \eta$.

auspices

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Definition 15.1. For any positive integer n and ordinals κ and η and abelian group A, an *n-auspicious family*

$$\Phi = \langle \varphi^{\varepsilon}_{\vec{\alpha}} : \alpha_0 \times \varepsilon \to A \mid (\vec{\alpha}, \varepsilon) \in [\kappa]^n \times \eta \rangle$$

is one exhibiting

(n.1) *horizontal n-coherence*, meaning that the families

$$\Phi^{\varepsilon} := \langle \varphi^{\varepsilon}_{\vec{\alpha}} : \alpha_0 \times \varepsilon \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$$

are n-coherent for all $\varepsilon < \eta$,

(n.2) *vertically persistent horizontal non-n-triviality*, meaning that for all $\delta < \eta$ there exists an $\varepsilon < \eta$ for which

$$\Phi^{\varepsilon} \restriction [\delta, \varepsilon) := \left\langle \varphi^{\varepsilon}_{\vec{\alpha}}\big|_{\alpha_0 \times [\delta, \varepsilon)} \mid \vec{\alpha} \in [\kappa]^n \right\rangle$$

is non-n-trivial, and

(n.3) *horizontally n-trivial vertical differences*, meaning that the families

$$\Phi^{\varepsilon} - \Phi^{\delta} := \left\langle \varphi^{\varepsilon}_{\vec{\alpha}} - \varphi^{\delta}_{\vec{\alpha}} : \alpha_0 \times \delta \to \mathbb{Z} \mid \vec{\alpha} \in [\kappa]^n \right\rangle$$

are n-trivial for all $\delta < \varepsilon < \eta$.

auspices

Definition 15.1. For any positive integer n and ordinals κ and η and abelian group A, an *n-auspicious family*

$$\Phi = \langle \varphi^{\varepsilon}_{\vec{\alpha}} : \alpha_0 \times \varepsilon \to A \mid (\vec{\alpha}, \varepsilon) \in [\kappa]^n \times \eta \rangle$$

is one exhibiting

(n.1) *horizontal n-coherence*, meaning that the families

$$\Phi^{\varepsilon} := \langle \varphi^{\varepsilon}_{\vec{\alpha}} : \alpha_0 \times \varepsilon \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$$

are n-coherent for all $\varepsilon < \eta$,

(n.2) *vertically persistent horizontal non-n-triviality*, meaning that for all $\delta < \eta$ there exists an $\varepsilon < \eta$ for which

$$\Phi^{\varepsilon} \restriction [\delta, \varepsilon) := \left\langle \varphi^{\varepsilon}_{\vec{\alpha}} \big|_{\alpha_0 \times [\delta, \varepsilon)} \mid \vec{\alpha} \in [\kappa]^n \right\rangle$$

is non-n-trivial, and

(n.3) *horizontally n-trivial vertical differences*, meaning that the families

$$\Phi^{\varepsilon} - \Phi^{\delta} := \left\langle \varphi^{\varepsilon}_{\vec{\alpha}} - \varphi^{\delta}_{\vec{\alpha}} : \alpha_0 \times \delta \to \mathbb{Z} \mid \vec{\alpha} \in [\kappa]^n \right\rangle$$

are n-trivial for all $\delta < \varepsilon < \eta$.

We say that such a Φ is of *width* κ and *height* η, and if κ is a cardinal and $\eta = \kappa^+$ then we call such a family a (κ, n)-*auspice*, for short.

auspice facts

auspice facts

THEOREM 15.3. *For every* $n > 0$ *and nontrivial abelian group* A *there exists an* A*-valued* (ω_n, n)*-auspice* Φ.

auspice facts

THEOREM 15.3. *For every $n > 0$ and nontrivial abelian group A there exists an A-valued (ω_n, n)-auspice Φ.*

THEOREM 15.5. *For every $n > 0$ and nontrivial abelian group A,* $\mathrm{H}^q(\omega_{n+1}; \mathcal{A}) \neq 0$ *for some* $0 < q \leq n+1$.

auspice facts

Theorem 15.3. *For every $n > 0$ and nontrivial abelian group A there exists an A-valued (ω_n, n)-auspice Φ.*

Theorem 15.5. *For every $n > 0$ and nontrivial abelian group A, $\mathrm{H}^q(\omega_{n+1}; \mathcal{A}) \neq 0$ for some $0 < q \leq n+1$.*

Corollary 15.6. *For any $n > 0$ and nontrivial abelian group A, the cohomology group $\mathrm{H}^n(\omega_k; \mathcal{A})$ is nonzero in the Eskew-Hayut model mentioned above if and only if $k = n$; therein, in particular, $\mathrm{H}^n(\omega_n; \mathbb{Z}) \neq 0$ for every $n \geq 0$.*

how an $(\omega_2, 2)$-auspice works

how an $(\omega_2, 2)$-auspice works

Example 15.7. *An A-valued $(\kappa, 1)$-auspice Φ is a family of families*

$$\Phi^\gamma = \langle \varphi^\gamma_\alpha : \alpha \times \gamma \to A \mid \alpha < \kappa \rangle \qquad (\gamma < \kappa^+)$$

which:

- *(1.1) are* 1-coherent*;*
- *(1.2) are* persistently non-1-trivial*: for all $\gamma < \kappa^+$ there exists a $\delta < \kappa^+$ so that $\Phi^\delta \restriction [\gamma, \delta)$ is non-1-trivial;*
- *(1.3)* differ 1-trivially*, in the sense that any $(\Phi^\delta - \Phi^\gamma) \restriction \kappa \times \gamma$ admits a trivialization $\psi^{\gamma\delta} : \kappa \times \gamma \to A$.*

how an $(\omega_2, 2)$-auspice works

Example 15.7. *An A-valued $(\kappa, 1)$-auspice Φ is a family of families*

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- *(1.3)* differ 1-trivially, *in the sense that any $(\Phi^\delta - \Phi^\gamma) \restriction \kappa \times \gamma$ admits a trivialization $\psi^{\gamma\delta} : \kappa \times \gamma \to A$.*

Fix such a $(\kappa, 1)$-auspice Φ and observe that the family

$$\Psi = \left\langle \psi^{\gamma\delta} : \kappa \times \gamma \to A \mid \gamma < \delta < \kappa^+ \right\rangle$$

of 1-trivializations of $\Phi^\delta - \Phi^\gamma$ is 2-coherent in the sense that any $\psi^{\delta\varepsilon} - \psi^{\gamma\varepsilon} + \psi^{\gamma\delta}$ is finitely supported. This follows immediately from our definitions: condition (1.2) ensures that κ is of uncountable cofinality, and for any $\alpha < \kappa$,

(32)
$$\left(\psi^{\delta\varepsilon} - \psi^{\gamma\varepsilon} + \psi^{\gamma\delta}\right)\big|_{\alpha\times\gamma} =^* \left(\left(\varphi^\varepsilon_\alpha - \varphi^\delta_\alpha\right) - \left(\varphi^\varepsilon_\alpha - \varphi^\gamma_\alpha\right) + \left(\varphi^\delta_\alpha - \varphi^\gamma_\alpha\right)\right)\big|_{\alpha\times\gamma} =^* 0.$$

Thus $\mathrm{H}^2(\kappa^+; \mathcal{A}) = 0$ implies, by Lemma 6.2, that Ψ admits a 2-trivialization

$$\Theta = \left\langle \theta^\gamma : \kappa \times \gamma \to A \mid \gamma < \kappa^+ \right\rangle.$$

But then

$$\Phi - \Theta := \left\langle \varphi^\gamma_\alpha - \theta^\gamma : \alpha \times \gamma \to A \mid (\alpha, \gamma) \in \kappa \times \kappa^+ \right\rangle$$

is a bicoherent system*, and this implies that $\mathrm{H}^1(\kappa^+; \mathcal{A}) \neq 0$.*

the moment of truth

the moment of truth

It's time to vote:

the moment of truth

It's time to vote:

Question

Is $\mathrm{H}^2(\omega_2; \mathbb{Z}) \neq 0$ *a* ZFC *theorem?*

May our votes all carry the force of theorems.

May our votes all carry the force of theorems.

Many many thanks to the organizers for the invitation

and to the audience for your attention

and for any questions which you may have.