The cohomology of the ordinals
day 2

Jeffrey Bergfalk
University of Barcelona

Winter School 2026

Hejnice, Czech Republic
February 2026

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is (1-)coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$.

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is (1-)coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$. Φ is trivial if there exists a $\psi : \kappa \to \alpha$ such that

$$\psi|_\alpha = \varphi_\alpha \pmod{\text{finite}}$$

for all $\alpha < \kappa$.

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is (1-)coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$. Φ is trivial if there exists a $\psi : \kappa \to \alpha$ such that

$$\psi|_\alpha = \varphi_\alpha \pmod{\text{finite}}$$

for all $\alpha < \kappa$.

Recall from yesterday:

Fact

$H^1(\kappa; \mathcal{A}) \neq 0$ *iff there exists a nontrivial coherent family (an NTC) of height* κ *and codomain* A.

2

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \beta < \kappa \rangle$ is 2-coherent if

$$\varphi_{\beta\gamma}|_\alpha - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \quad \text{(mod finite)}$$

for all $\alpha < \beta < \gamma < \kappa$.

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \beta < \kappa \rangle$ is 2-coherent if

$$\varphi_{\beta\gamma}|_\alpha - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \quad \text{(mod finite)}$$

for all $\alpha < \beta < \gamma < \kappa$. Φ is trivial if there exists a family $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\psi_\beta|_\alpha - \psi_\alpha = \varphi_{\alpha\beta} \quad \text{(mod finite)}$$

for all $\alpha < \beta < \kappa$.

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \beta < \kappa \rangle$ is 2-coherent if

$$\varphi_{\beta\gamma}|_\alpha - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \quad \text{(mod finite)}$$

for all $\alpha < \beta < \gamma < \kappa$. Φ is trivial if there exists a family $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\psi_\beta|_\alpha - \psi_\alpha = \varphi_{\alpha\beta} \quad \text{(mod finite)}$$

for all $\alpha < \beta < \kappa$.

Again from yesterday:

Fact

$\mathrm{H}^2(\kappa; \mathcal{A}) \neq 0$ iff there exists a nontrivial 2-coherent family (an NT2C) of height κ and codomain A.

$$n > 2$$

$n > 2$

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$ is n-coherent if

$$\sum_{i=0}^{n} (-1)^i \varphi_{\vec{\alpha}^i} = 0 \pmod{\text{finite}}$$

for all $\vec{\alpha} \in [\kappa]^{n+1}$.

$n > 2$

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$ is n-coherent if

$$\sum_{i=0}^{n} (-1)^i \varphi_{\vec{\alpha}^i} = 0 \pmod{\text{finite}}$$

for all $\vec{\alpha} \in [\kappa]^{n+1}$. Φ is trivial if there exists a $\Psi = \langle \psi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^{n-1} \rangle$ such that

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{\alpha}^i} = \varphi_{\vec{\alpha}} \pmod{\text{finite}}$$

for all $\vec{\alpha} \in [\kappa]^n$.

$n > 2$

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$ is n-coherent if

$$\sum_{i=0}^{n} (-1)^i \varphi_{\vec{\alpha}^i} = 0 \pmod{\text{finite}}$$

for all $\vec{\alpha} \in [\kappa]^{n+1}$. Φ is trivial if there exists a $\Psi = \langle \psi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^{n-1} \rangle$ such that

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{\alpha}^i} = \varphi_{\vec{\alpha}} \pmod{\text{finite}}$$

for all $\vec{\alpha} \in [\kappa]^n$.

Fact

$\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ *iff there exists a nontrivial n-coherent family (an NTnC) of height κ and codomain A.*

above the diagonal

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week.

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial $\mathit{2}$-coherence, and so on, prior to this week. Why is this?

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week. Why is this?

One reason is the following variant of *Goblot's Theorem* (1970):

Theorem

For all $0 \leq m < n$ *and abelian groups* A *and ordinals* κ, *if* $\mathrm{cf}(\kappa) \leq \omega_m$ *then* $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week. Why is this?

One reason is the following variant of *Goblot's Theorem* (1970):

Theorem

For all $0 \leq m < n$ *and abelian groups* A *and ordinals* κ, *if* $\mathrm{cf}(\kappa) \leq \omega_m$ *then* $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

Exercise

1. Prove the $n = 1$ *instance of the theorem.*

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week. Why is this?

One reason is the following variant of *Goblot's Theorem* (1970):

Theorem

For all $0 \leq m < n$ *and abelian groups* A *and ordinals* κ*, if* $\mathrm{cf}(\kappa) \leq \omega_m$ *then* $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

Exercise

1. Prove the $n = 1$ *instance of the theorem.*

Show, in other words, that if κ is of countable cofinality then every coherent family of height κ is trivial.

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week. Why is this?

One reason is the following variant of *Goblot's Theorem* (1970):

Theorem

For all $0 \leq m < n$ *and abelian groups* A *and ordinals* κ, *if* $\mathrm{cf}(\kappa) \leq \omega_m$ *then* $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

Exercise

1. Prove the $n = 1$ *instance of the theorem.*

Show, in other words, that if κ is of countable cofinality then every coherent family of height κ is trivial. More generally, the theorem's telling us that ω_n is the least κ on which a nontrivial n-coherent family can conceivably appear:

above the diagonal

While many of you will have encountered nontrivial coherence, fewer will have encountered nontrivial *2*-coherence, and so on, prior to this week. Why is this?

One reason is the following variant of *Goblot's Theorem* (1970):

Theorem

For all $0 \leq m < n$ *and abelian groups* A *and ordinals* κ*, if* $\mathrm{cf}(\kappa) \leq \omega_m$ *then* $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

Exercise

1. Prove the $n = 1$ *instance of the theorem.*

Show, in other words, that if κ is of countable cofinality then every coherent family of height κ is trivial. More generally, the theorem's telling us that ω_n is the least κ on which a nontrivial n-coherent family can conceivably appear: above the (ω_n, H^n)-diagonal, it fills our chart with zeros.

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*)

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*) (thus we may read any $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ as an *incompactness principle*, asserting the existence of height-κ objects whose small- and large-scale behaviors sharply diverge).

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*) (thus we may read any $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ as an *incompactness principle*, asserting the existence of height-κ objects whose small- and large-scale behaviors sharply diverge). Conversely, an n-coherent Φ is trivial if and only if it properly extends to a taller n-coherent family (*exercise 2'*).

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*) (thus we may read any $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ as an *incompactness principle*, asserting the existence of height-κ objects whose small- and large-scale behaviors sharply diverge). Conversely, an n-coherent Φ is trivial if and only if it properly extends to a taller n-coherent family (*exercise 2'*).

Speaking of extensions:

Exercise

3. For any $E \subseteq \kappa$ and $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$, write $\Phi \restriction E$ for $\langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [E]^n \rangle$. Show that if E is cofinal in κ then

- *Φ is nontrivially n-coherent if and only if $\Phi \restriction E$ is*

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*) (thus we may read any $\mathrm{H}^n(\kappa;\mathcal{A}) \neq 0$ as an *incompactness principle*, asserting the existence of height-κ objects whose small- and large-scale behaviors sharply diverge). Conversely, an n-coherent Φ is trivial if and only if it properly extends to a taller n-coherent family (*exercise 2'*).

Speaking of extensions:

Exercise

3. For any $E \subseteq \kappa$ and $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$, write $\Phi \restriction E$ for $\langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [E]^n \rangle$. Show that if E is cofinal in κ then

- *Φ is nontrivially n-coherent if and only if $\Phi \restriction E$ is, and*
- *that any n-coherent $\langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [E]^n \rangle$ n-coherently extends to a Φ as above.*

Note next that any proper initial segment of an n-coherent Φ is trivial (*exercise 2*) (thus we may read any $\mathrm{H}^n(\kappa;\mathcal{A}) \neq 0$ as an *incompactness principle*, asserting the existence of height-κ objects whose small- and large-scale behaviors sharply diverge). Conversely, an n-coherent Φ is trivial if and only if it properly extends to a taller n-coherent family (*exercise 2'*).

Speaking of extensions:

Exercise

3. For any $E \subseteq \kappa$ and $\Phi = \langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [\kappa]^n \rangle$, write $\Phi \restriction E$ for $\langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [E]^n \rangle$. Show that if E is cofinal in κ then

- *Φ is nontrivially n-coherent if and only if $\Phi \restriction E$ is, and*
- *that any n-coherent $\langle \varphi_{\vec{\alpha}} : \alpha_0 \to A \mid \vec{\alpha} \in [E]^n \rangle$ n-coherently extends to a Φ as above.*

Deduce that if $\mathrm{H}^n(\kappa;\mathcal{A}) \neq 0$ and κ cofinally order-embeds into λ then $\mathrm{H}^n(\lambda;\mathcal{A}) \neq 0$ as well.

further heuristics

further heuristics

We could see nontrivial n-coherence

further heuristics

We could see nontrivial n-coherence

- as a question of whether local solutions can be globalized;

further heuristics

We could see nontrivial n-coherence

- as a question of whether local solutions can be globalized;
- in terms of rigidity of dimension;

further heuristics

We could see nontrivial n-coherence

- as a question of whether local solutions can be globalized;
- in terms of rigidity of dimension;
- in terms of trees;

further heuristics

We could see nontrivial n-coherence

- as a question of whether local solutions can be globalized;
- in terms of rigidity of dimension;
- in terms of trees;
- via slightly different formulations

Definition 7.8. A function is *0-trivial* if its support is finite.

For any $n > 0$ a family of functions $\{s_\gamma^n : [\gamma]^n \to A \mid \gamma < \varepsilon\}$ is *n-coherent* if

$$s_\delta^n\big|_{[\gamma]^n} - s_\gamma^n$$

is $(n-1)$-trivial for all $\gamma \leq \delta < \varepsilon$.

Such a family is *n-trivial* if there exists a $t^n : [\varepsilon]^n \to A$ such that

$$t^n\big|_{[\gamma]^n} - s_\gamma^n$$

is $(n-1)$-trivial for all $\gamma < \varepsilon$.

whose quotients *n-coherent*/*n-trivial* turn out to also equal $\mathrm{H}^n(\varepsilon; \mathcal{A})$.

Let's collect one more general-purpose observation in an

Exercise

Let's collect one more general-purpose observation in an

Exercise

4. Let $n > 1$. Show that if Θ and Ψ both n-trivialize an n-coherent family Φ then $\Theta - \Psi$ is $(n-1)$-coherent.

Let's collect one more general-purpose observation in an

Exercise

4. Let $n > 1$. Show that if Θ and Ψ both n-trivialize an n-coherent family Φ then $\Theta - \Psi$ is $(n-1)$-coherent.

This last observation plays a role in the proof of Goblot's vanishing theorem above.

Let's collect one more general-purpose observation in an

Exercise

4. Let $n > 1$. Show that if Θ and Ψ both n-trivialize an n-coherent family Φ then $\Theta - \Psi$ is $(n-1)$-coherent.

This last observation plays a role in the proof of Goblot's vanishing theorem above.

A main mechanism of many *nonvanishing* results is a closely related trick for *diagonalizing against potential trivializations* while we construct an n-coherent family.

Let's collect one more general-purpose observation in an

Exercise

4. Let $n > 1$. Show that if Θ and Ψ both n-trivialize an n-coherent family Φ then $\Theta - \Psi$ is $(n-1)$-coherent.

This last observation plays a role in the proof of Goblot's vanishing theorem above.

A main mechanism of many *nonvanishing* results is a closely related trick for *diagonalizing against potential trivializations* while we construct an n-coherent family. To see what I mean, let's apply the trick to construct a nontrivial 2-coherent family of height ω_2.

Let's collect one more general-purpose observation in an

Exercise

4. Let $n > 1$. Show that if Θ and Ψ both n-trivialize an n-coherent family Φ then $\Theta - \Psi$ is $(n-1)$-coherent.

This last observation plays a role in the proof of Goblot's vanishing theorem above.

A main mechanism of many *nonvanishing* results is a closely related trick for *diagonalizing against potential trivializations* while we construct an n-coherent family. To see what I mean, let's apply the trick to construct a nontrivial 2-coherent family of height ω_2. For maximal clarity, we'll do this with the aid of $\diamondsuit(S^2_1)$, which we will here read as asserting that

$$\text{there exists an } \mathcal{L} = \langle \Psi^\alpha \mid \alpha \in S^2_1 \rangle$$
$$\text{such that for any } \Psi = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_2 \rangle$$
$$\text{there's an } \alpha \in S^2_1 \text{ such that } \Psi \restriction \alpha = \Psi^\alpha.$$

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$.

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

Let's define a 2-coherent $\Phi \restriction \omega_1 + 1$ extension of $\Phi \restriction \omega_1$ in such a way that no extension of Ψ^{ω_1} trivializes it: simply let $\varphi_{\alpha\omega_1} = \theta_\alpha - \psi_\alpha$ for each $\alpha < \omega_1$.

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

Let's define a 2-coherent $\Phi \restriction \omega_1 + 1$ extension of $\Phi \restriction \omega_1$ in such a way that no extension of Ψ^{ω_1} trivializes it: simply let $\varphi_{\alpha\omega_1} = \theta_\alpha - \psi_\alpha$ for each $\alpha < \omega_1$. It's easy to check that $\Phi \restriction \omega_1 + 1$ is 2-coherent.

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

Let's define a 2-coherent $\Phi \restriction \omega_1 + 1$ extension of $\Phi \restriction \omega_1$ in such a way that no extension of Ψ^{ω_1} trivializes it: simply let $\varphi_{\alpha\omega_1} = \theta_\alpha - \psi_\alpha$ for each $\alpha < \omega_1$. It's easy to check that $\Phi \restriction \omega_1 + 1$ is 2-coherent. Next, suppose for contradiction that Ψ^{ω_1} extends to a trivialization of $\Phi \restriction \omega_1 + 1$, so that there exists a ψ_{ω_1} with $\psi_{\omega_1} - \psi_\alpha =^* \varphi_{\alpha\omega_1}$ for all $\alpha < \omega_1$.

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

Let's define a 2-coherent $\Phi \restriction \omega_1 + 1$ extension of $\Phi \restriction \omega_1$ in such a way that no extension of Ψ^{ω_1} trivializes it: simply let $\varphi_{\alpha\omega_1} = \theta_\alpha - \psi_\alpha$ for each $\alpha < \omega_1$. It's easy to check that $\Phi \restriction \omega_1 + 1$ is 2-coherent. Next, suppose for contradiction that Ψ^{ω_1} extends to a trivialization of $\Phi \restriction \omega_1 + 1$, so that there exists a ψ_{ω_1} with $\psi_{\omega_1} - \psi_\alpha =^* \varphi_{\alpha\omega_1}$ for all $\alpha < \omega_1$. But then

$$\psi_{\omega_1} =^* \varphi_{\alpha\omega_1} + \psi_\alpha = \theta_\alpha \text{ for all } \alpha < \omega_1$$

For simplicity, suppose that we've constructed

$$\Phi \restriction \omega_1 = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_1 \rangle$$

and that $\mathcal{L}$ hands us a

$$\Psi^{\omega_1} = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle$$

2-trivializing $\Phi \restriction \omega_1$. Fix a nontrivial coherent family

$$\Theta = \langle \theta_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \omega_1 \rangle.$$

Let's define a 2-coherent $\Phi \restriction \omega_1 + 1$ extension of $\Phi \restriction \omega_1$ in such a way that no extension of Ψ^{ω_1} trivializes it: simply let $\varphi_{\alpha\omega_1} = \theta_\alpha - \psi_\alpha$ for each $\alpha < \omega_1$. It's easy to check that $\Phi \restriction \omega_1 + 1$ is 2-coherent. Next, suppose for contradiction that Ψ^{ω_1} extends to a trivialization of $\Phi \restriction \omega_1 + 1$, so that there exists a ψ_{ω_1} with $\psi_{\omega_1} - \psi_\alpha =^* \varphi_{\alpha\omega_1}$ for all $\alpha < \omega_1$. But then

$$\psi_{\omega_1} =^* \varphi_{\alpha\omega_1} + \psi_\alpha = \theta_\alpha \text{ for all } \alpha < \omega_1 : \text{ contradiction.}$$

It's *exercise 5* to complete the proof of the following theorem.

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ *holds, then* $\mathrm{H}^2(\omega_2; \mathbb{Z}) \neq 0$.

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ *holds, then* $\mathrm{H}^2(\omega_2;\mathbb{Z}) \neq 0$*. If* $\diamondsuit(S^2_1)$ *and* $\diamondsuit(S^3_2)$ *hold, then* $\mathrm{H}^3(\omega_3;\mathbb{Z}) \neq 0$

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ *holds, then* $\mathrm{H}^2(\omega_2;\mathbb{Z}) \neq 0$. *If* $\diamondsuit(S^2_1)$ *and* $\diamondsuit(S^3_2)$ *hold, then* $\mathrm{H}^3(\omega_3;\mathbb{Z}) \neq 0$ *— and so on.*

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ holds, then $\mathrm{H}^2(\omega_2;\mathbb{Z}) \neq 0$. If $\diamondsuit(S^2_1)$ and $\diamondsuit(S^3_2)$ hold, then $\mathrm{H}^3(\omega_3;\mathbb{Z}) \neq 0$ — and so on.

Q: What new issue arises when we attempt this construction on, say, ω_3, using $\diamondsuit(S^3_1)$?

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ holds, then $\mathrm{H}^2(\omega_2;\mathbb{Z})\neq 0$. If $\diamondsuit(S^2_1)$ and $\diamondsuit(S^3_2)$ hold, then $\mathrm{H}^3(\omega_3;\mathbb{Z})\neq 0$ — and so on.

Q: What new issue arises when we attempt this construction on, say, ω_3, using $\diamondsuit(S^3_1)$?
A: We could get stuck below ω_3.

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ holds, then $\mathrm{H}^2(\omega_2;\mathbb{Z}) \neq 0$. If $\diamondsuit(S^2_1)$ and $\diamondsuit(S^3_2)$ hold, then $\mathrm{H}^3(\omega_3;\mathbb{Z}) \neq 0$ — and so on.

Q: What new issue arises when we attempt this construction on, say, ω_3, using $\diamondsuit(S^3_1)$?
A: We could get stuck below ω_3.

Thus:

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ *holds, then* $\mathrm{H}^2(\omega_2;\mathbb{Z})\neq 0$. *If* $\diamondsuit(S^2_1)$ *and* $\diamondsuit(S^3_2)$ *hold, then* $\mathrm{H}^3(\omega_3;\mathbb{Z})\neq 0$ *— and so on.*

Q: What new issue arises when we attempt this construction on, say, ω_3, using $\diamondsuit(S^3_1)$?
A: We could get stuck below ω_3.

Thus:
Q: Can there exist nontrivial 2-coherent families of height ω_3?

It's *exercise 5* to complete the proof of the following theorem.

Theorem

If $\diamondsuit(S^2_1)$ *holds, then* $\mathrm{H}^2(\omega_2;\mathbb{Z}) \neq 0$*. If* $\diamondsuit(S^2_1)$ *and* $\diamondsuit(S^3_2)$ *hold, then* $\mathrm{H}^3(\omega_3;\mathbb{Z}) \neq 0$ *— and so on.*

Q: What new issue arises when we attempt this construction on, say, ω_3, using $\diamondsuit(S^3_1)$?
A: We could get stuck below ω_3.

Thus:
Q: Can there exist nontrivial 2-coherent families of height ω_3?
A: Definitely.

forcing nontrivial coherence

forcing nontrivial coherence

Theorem

Let A be a nonzero abelian group. Suppose either that $n = 1$ and $\kappa = \omega$ or that $n > 1$ and $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ for some κ. Then for any regular $\lambda > \kappa$ there exists a λ-strategically closed forcing notion $\mathbb{P}(n, \lambda, A)$ which adds a witness to $\mathrm{H}^n(\lambda; \mathcal{A}) \neq 0$.

forcing nontrivial coherence

Theorem

Let A be a nonzero abelian group. Suppose either that $n = 1$ and $\kappa = \omega$ or that $n > 1$ and $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ for some κ. Then for any regular $\lambda > \kappa$ there exists a λ-strategically closed forcing notion $\mathbb{P}(n, \lambda, A)$ which adds a witness to $\mathrm{H}^n(\lambda; \mathcal{A}) \neq 0$.

The conditions of $\mathbb{P}(n, \lambda, A)$ are A-valued n-coherent families of successor height below λ, and the ordering is reverse-inclusion; the aforementioned witness is simply $\bigcup G$ for a $\mathbb{P}(n, \lambda, A)$-generic filter G.

forcing nontrivial coherence

Theorem

Let A be a nonzero abelian group. Suppose either that $n = 1$ and $\kappa = \omega$ or that $n > 1$ and $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ for some κ. Then for any regular $\lambda > \kappa$ there exists a λ-strategically closed forcing notion $\mathbb{P}(n, \lambda, A)$ which adds a witness to $\mathrm{H}^n(\lambda; \mathcal{A}) \neq 0$.

The conditions of $\mathbb{P}(n, \lambda, A)$ are A-valued n-coherent families of successor height below λ, and the ordering is reverse-inclusion; the aforementioned witness is simply $\bigcup G$ for a $\mathbb{P}(n, \lambda, A)$-generic filter G. Since $\mathbb{P}(n, \lambda, A)$ is λ-strategically closed, it adds no sequences of length less than λ of elements of V; in particular, it preserves all cardinals and cofinalities less than or equal to λ.

forcing nontrivial coherence

Theorem

Let A be a nonzero abelian group. Suppose either that $n = 1$ and $\kappa = \omega$ or that $n > 1$ and $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ for some κ. Then for any regular $\lambda > \kappa$ there exists a λ-strategically closed forcing notion $\mathbb{P}(n, \lambda, A)$ which adds a witness to $\mathrm{H}^n(\lambda; \mathcal{A}) \neq 0$.

The conditions of $\mathbb{P}(n, \lambda, A)$ are A-valued n-coherent families of successor height below λ, and the ordering is reverse-inclusion; the aforementioned witness is simply $\bigcup G$ for a $\mathbb{P}(n, \lambda, A)$-generic filter G. Since $\mathbb{P}(n, \lambda, A)$ is λ-strategically closed, it adds no sequences of length less than λ of elements of V; in particular, it preserves all cardinals and cofinalities less than or equal to λ. Only two points require much thought here: the first is the strategy which secures strategic closure, and the second is the argument that $\bigcup G$ is nontrivial; the latter recycles the trick we used with $\diamondsuit$ above.

nontrivial coherence everywhere

nontrivial coherence everywhere

If one would prefer not to force, one can still get NTnCs on $\lambda > \omega_n$ from $\diamondsuit$s provided one has ways of ensuring that one's construction applying them doesn't terminate too early.

nontrivial coherence everywhere

If one would prefer not to force, one can still get NTnCs on $\lambda > \omega_n$ from $\diamondsuit$s provided one has ways of ensuring that one's construction applying them doesn't terminate too early.

Theorem

Let A be a nonzero abelian group and let $\kappa < \lambda$ be infinite regular cardinals. Suppose either that

- $n = 1$ *and* $\square(\lambda)$ *holds, or that*
- $n < 1$ *and* $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ *and both* $\diamondsuit(S)$ *and* $\square(\lambda, S)$ *hold for some stationary* $S \subseteq S^\lambda_\kappa$.

Then $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

nontrivial coherence everywhere

If one would prefer not to force, one can still get NTnCs on $\lambda > \omega_n$ from $\diamondsuit$s provided one has ways of ensuring that one's construction applying them doesn't terminate too early.

Theorem

Let A be a nonzero abelian group and let $\kappa < \lambda$ be infinite regular cardinals. Suppose either that

- $n = 1$ *and* $\square(\lambda)$ *holds, or that*
- $n < 1$ *and* $\mathrm{H}^{n-1}(\kappa; \mathcal{A}) \neq 0$ *and both* $\diamondsuit(S)$ *and* $\square(\lambda, S)$ *hold for some stationary* $S \subseteq S^\lambda_\kappa$.

Then $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$.

Corollary

In a model of "$V = L$ and there exist no weakly compact cardinals" $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ *in every instance not precluded by Goblot's theorem.*

The corollary further certifies nonvanishing H^n $(n > 0)$ as a *graded family of incompactness principles.*

The corollary further certifies nonvanishing H^n $(n > 0)$ as a *graded family of incompactness principles.* These principles are, moreover, allergic to large cardinals for much the same reason that Aronszajn trees are (at least for small coefficient groups A; they're still alergic for larger A, but this seems to require more sophisticated argument).

The corollary further certifies nonvanishing H^n $(n > 0)$ as a *graded family of incompactness principles.* These principles are, moreover, allergic to large cardinals for much the same reason that Aronszajn trees are (at least for small coefficient groups A; they're still alergic for larger A, but this seems to require more sophisticated argument). A natural strategy for annulling these principles on smaller cardinals, then, is via large cardinal collapses, but there are interesting limitations to this approach.

The corollary further certifies nonvanishing H^n $(n > 0)$ as a *graded family of incompactness principles*. These principles are, moreover, allergic to large cardinals for much the same reason that Aronszajn trees are (at least for small coefficient groups A; they're still alergic for larger A, but this seems to require more sophisticated argument). A natural strategy for annulling these principles on smaller cardinals, then, is via large cardinal collapses, but there are interesting limitations to this approach.

Theorem

If κ is a weakly compact cardinal then $\mathrm{H}^1(\omega_2; \mathcal{A}) = 0$ but $\mathrm{H}^2(\omega_2; \mathcal{A}) \neq 0$ for every nonzero abelian group A in the forcing extension by $\mathrm{Coll}(\omega_1, < \kappa)$.

The corollary further certifies nonvanishing H^n $(n > 0)$ as a *graded family of incompactness principles*. These principles are, moreover, allergic to large cardinals for much the same reason that Aronszajn trees are (at least for small coefficient groups A; they're still alergic for larger A, but this seems to require more sophisticated argument). A natural strategy for annulling these principles on smaller cardinals, then, is via large cardinal collapses, but there are interesting limitations to this approach.

Theorem

If κ is a weakly compact cardinal then $\mathrm{H}^1(\omega_2; \mathcal{A}) = 0$ but $\mathrm{H}^2(\omega_2; \mathcal{A}) \neq 0$ for every nonzero abelian group A in the forcing extension by $\mathrm{Coll}(\omega_1, < \kappa)$.

See you tomorrow!