The cohomology of the ordinals

Jeffrey Bergfalk
University of Barcelona

Winter School 2026

Hejnice, Czech Republic
February 2026

background

background

In recent years, a major area of interaction between set theory and more general mathematics has been *the infinitary combinatorics of various (co)homology groups* of degree $n > 1$.

background

In recent years, a major area of interaction between set theory and more general mathematics has been *the infinitary combinatorics of various (co)homology groups* of degree $n > 1$.

Questions in this area often reduce (or conjecturally reduce) to questions about

the cohomology groups of the ordinals

themselves.

background

In recent years, a major area of interaction between set theory and more general mathematics has been *the infinitary combinatorics of various (co)homology groups* of degree $n > 1$.

Questions in this area often reduce (or conjecturally reduce) to questions about

the cohomology groups of the ordinals

themselves.

There's something about these latter groups that seems to call for new ways of thinking, and it is this seeming necessity, most broadly, which is the subject of this series of talks.

the plan

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena;

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material.

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence, and second, that almost everything I discuss in these talks is joint work with both Chris and Jing Zhang.

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence, and second, that almost everything I discuss in these talks is joint work with both Chris and Jing Zhang. Tomorrow we'll discuss the sensitivity of these groups' values to the universe they're computed in.

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence, and second, that almost everything I discuss in these talks is joint work with both Chris and Jing Zhang. Tomorrow we'll discuss the sensitivity of these groups' values to the universe they're computed in. Friday we'll describe a few combinatorial principles (*auspices*, *bicoherent systems*, *partition hypotheses*) first uncovered in the course of this research

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence, and second, that almost everything I discuss in these talks is joint work with both Chris and Jing Zhang. Tomorrow we'll discuss the sensitivity of these groups' values to the universe they're computed in. Friday we'll describe a few combinatorial principles (*auspices*, *bicoherent systems*, *partition hypotheses*) first uncovered in the course of this research and resolve the

Question

Is $\mathrm{H}^2(\omega_2; \mathbb{Z}) \neq 0$ *a* ZFC *theorem?*

the plan

I'll propose cohomology as a powerfully unifying language and tool for the study of both classical and higher-dimensional coherence phenomena; since the latter was the subject of Chris Lambie-Hanson's WS2022 tutorial, veterans of that Winter School got a bit of a head start on this material. I mention this to say two things: first, that Chris's slides are an excellent resource for those looking for more on today's theme of cohomology and coherence, and second, that almost everything I discuss in these talks is joint work with both Chris and Jing Zhang. Tomorrow we'll discuss the sensitivity of these groups' values to the universe they're computed in. Friday we'll describe a few combinatorial principles (*auspices*, *bicoherent systems*, *partition hypotheses*) first uncovered in the course of this research and resolve the

Question

Is $\mathrm{H}^2(\omega_2; \mathbb{Z}) \neq 0$ *a* ZFC *theorem?*

with a vote.

ω_1

ω_1

When I say things like *the combinatorics of* κ I mean: *the behavior of classes of objects of size at most* κ or *of objects naturally encoded by subsets of* κ, things like this.

ω_1

When I say things like *the combinatorics of* κ I mean: *the behavior of classes of objects of size at most* κ or *of objects naturally encoded by subsets of* κ, things like this.

Example

A milestone in our understanding of the combinatorics of ω_1 was Justin Moore's proof that

the Proper Forcing Axiom implies that there exists a five-element basis for the class of uncountable linear orders

ω_1

When I say things like *the combinatorics of* κ I mean: *the behavior of classes of objects of size at most* κ or *of objects naturally encoded by subsets of* κ, things like this.

Example

A milestone in our understanding of the combinatorics of ω_1 was Justin Moore's proof that

the Proper Forcing Axiom implies that there exists a five-element basis for the class of uncountable linear orders

This is one of a family of results which couple, on the one side, a list of *critical* ZFC *objects of size* ω_1 with, on the other side, *a forcing axiom* showing that this list is complete

ω_1

When I say things like *the combinatorics of* κ I mean: *the behavior of classes of objects of size at most* κ or *of objects naturally encoded by subsets of* κ, things like this.

Example

A milestone in our understanding of the combinatorics of ω_1 was Justin Moore's proof that

the Proper Forcing Axiom implies that there exists a five-element basis for the class of uncountable linear orders

This is one of a family of results which couple, on the one side, a list of *critical* ZFC *objects of size* ω_1 with, on the other side, *a forcing axiom* showing that this list is complete — and which together constitute one of the major successes of the field.

ω_1

ω_1

The enumeration of these critical objects of size ω_1 passes, with a striking frequency, through

- a *coherent Aronszajn tree*

ω_1

The enumeration of these critical objects of size ω_1 passes, with a striking frequency, through

- a *coherent Aronszajn tree*, or, equivalently,
- a *nontrivial coherent family of functions*

ω_1

The enumeration of these critical objects of size ω_1 passes, with a striking frequency, through

- a *coherent Aronszajn tree*, or, equivalently,
- a *nontrivial coherent family of functions*, or, equivalently,
- *the first sheaf cohomology group* $\mathrm{H}^1(\omega_1; \mathbb{Z})$ *of* ω_1.

ω_1

The enumeration of these critical objects of size ω_1 passes, with a striking frequency, through

- a *coherent Aronszajn tree*, or, equivalently,
- a *nontrivial coherent family of functions*, or, equivalently,
- *the first sheaf cohomology group* $\mathrm{H}^1(\omega_1; \mathbb{Z})$ *of* ω_1.

A first goal for today is to explain these equivalences

ω_1

The enumeration of these critical objects of size ω_1 passes, with a striking frequency, through

- a *coherent Aronszajn tree*, or, equivalently,
- a *nontrivial coherent family of functions*, or, equivalently,
- *the first sheaf cohomology group* $\mathrm{H}^1(\omega_1; \mathbb{Z})$ *of* ω_1.

A first goal for today is to explain these equivalences, in part because they're pretty — but more importantly because of how they help us to think about what of *the* $\kappa = \omega_1$ *template for understanding the combinatorics of* κ may or may not extend to higher cardinals κ.

[Tuesday night edit]

[Tuesday night edit]

(Walks on the ordinals are an integral part of this story, but I'd planned to skip it, figuring people are tired of hearing me talk about them.

[Tuesday night edit]

(Walks on the ordinals are an integral part of this story, but I'd planned to skip it, figuring people are tired of hearing me talk about them.

Dramatic recent events, though, have led me to question this plan.

[Tuesday night edit]

(Walks on the ordinals are an integral part of this story, but I'd planned to skip it, figuring people are tired of hearing me talk about them.

Dramatic recent events, though, have led me to question this plan.

We'll play it by ear: maybe now, maybe later, it may be a good idea to at least briefly walk on some countable ordinals.

[Tuesday night edit]

(Walks on the ordinals are an integral part of this story, but I'd planned to skip it, figuring people are tired of hearing me talk about them.

Dramatic recent events, though, have led me to question this plan.

We'll play it by ear: maybe now, maybe later, it may be a good idea to at least briefly walk on some countable ordinals.

The key point for our purposes in any case is that walks are a main source of the coherent families of functions of the previous and following slides.)

coherence

coherence

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$.

coherence

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$. Φ is trivial if there exists a $\psi : \kappa \to \alpha$ such that

$$\psi|_\alpha = \varphi_\alpha \pmod{\text{finite}}$$

for all $\alpha < \kappa$.

coherence

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \pmod{\text{finite}}$$

for all $\alpha < \beta < \kappa$. Φ is trivial if there exists a $\psi : \kappa \to \alpha$ such that

$$\psi|_\alpha = \varphi_\alpha \pmod{\text{finite}}$$

for all $\alpha < \kappa$.

Todorcevic's *walks* techniques are a main way to see the following for any nonzero abelian group A:

Fact

There exists a nontrivial coherent Φ as above, of height $\kappa = \omega_1$.

coherence

Definition

For any ordinal κ and abelian group A, a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ is coherent if

$$\varphi_\beta|_\alpha - \varphi_\alpha = 0 \quad (\text{mod finite})$$

for all $\alpha < \beta < \kappa$. Φ is trivial if there exists a $\psi : \kappa \to \alpha$ such that

$$\psi|_\alpha = \varphi_\alpha \quad (\text{mod finite})$$

for all $\alpha < \kappa$.

Todorcevic's *walks* techniques are a main way to see the following for any nonzero abelian group A:

Fact

There exists a nontrivial coherent Φ as above, of height $\kappa = \omega_1$.

Such families efficiently give rise to ω_1-Aronszajn trees and to many of the other critical objects on ω_1 alluded to above.

two views

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A, namely the *modulus*: walks induce nontrivial coherent families *modulo continuous functions*, and in several other senses as well.

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A, namely the *modulus*: walks induce nontrivial coherent families *modulo continuous functions*, and in several other senses as well.

By a certain ambidexterity of the Von Neumann construction of the ordinals, we may view any such family as indexed either by

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A, namely the *modulus*: walks induce nontrivial coherent families *modulo continuous functions*, and in several other senses as well.

By a certain ambidexterity of the Von Neumann construction of the ordinals, we may view any such family as indexed either by

1. *elements of the partial order* κ

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A, namely the *modulus*: walks induce nontrivial coherent families *modulo continuous functions*, and in several other senses as well.

By a certain ambidexterity of the Von Neumann construction of the ordinals, we may view any such family as indexed either by

1. *elements of the partial order* κ, or by
2. *open subsets*, and in fact a *cover*, of *the topological space* κ.

two views

We'll take the preceding definition as our main one, but should note a third parameter in play along with κ and A, namely the *modulus*: walks induce nontrivial coherent families *modulo continuous functions*, and in several other senses as well.

By a certain ambidexterity of the Von Neumann construction of the ordinals, we may view any such family as indexed either by

1. *elements of the partial order* κ, or by
2. *open subsets*, and in fact a *cover*, of *the topological space* κ.

In either view, these families take their place in a larger network of groups and group-systems, to which we now turn.

the first view

the first view

Definition

An *inverse system of abelian groups on* κ is a family of abelian groups G_α ($\alpha < \kappa$) linked by a family of *transition* or *bonding maps*

$$\pi_{\alpha\beta} : G_\beta \to G_\alpha \qquad (\alpha \le \beta < \kappa)$$

satisfying $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$ for all $\alpha \le \beta \le \gamma < \kappa$.

the first view

Definition

An *inverse system of abelian groups on* κ is a family of abelian groups G_α $(\alpha < \kappa)$ linked by a family of *transition* or *bonding maps*

$$\pi_{\alpha\beta} : G_\beta \to G_\alpha \qquad (\alpha \leq \beta < \kappa)$$

satisfying $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma < \kappa$.
We sometimes denote such systems $\mathbf{G} = (G_\alpha, \pi_{\alpha\beta}, \kappa)$.

the first view

Definition

An *inverse system of abelian groups on* κ is a family of abelian groups G_α ($\alpha < \kappa$) linked by a family of *transition* or *bonding maps*

$$\pi_{\alpha\beta} : G_\beta \to G_\alpha \qquad (\alpha \leq \beta < \kappa)$$

satisfying $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma < \kappa$.
We sometimes denote such systems $\mathbf{G} = (G_\alpha, \pi_{\alpha\beta}, \kappa)$.

More concisely, such a $\mathbf{G}$ is a *contravariant functor from* κ *to* Ab; still more concisely, it is an Ab-*valued presheaf on* κ.

the first view

Definition

An *inverse system of abelian groups on* κ is a family of abelian groups G_α ($\alpha < \kappa$) linked by a family of *transition* or *bonding maps*

$$\pi_{\alpha\beta} : G_\beta \to G_\alpha \qquad (\alpha \leq \beta < \kappa)$$

satisfying $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma < \kappa$.
We sometimes denote such systems $\mathbf{G} = (G_\alpha, \pi_{\alpha\beta}, \kappa)$.

More concisely, such a $\mathbf{G}$ is a *contravariant functor from* κ *to* Ab; still more concisely, it is an Ab-*valued presheaf on* κ. Write $\mathsf{Pr}(\kappa, \mathsf{Ab})$ for the category of such systems (the morphisms therein are natural transformations),

the first view

Definition

An *inverse system of abelian groups on* κ is a family of abelian groups G_α $(\alpha < \kappa)$ linked by a family of *transition* or *bonding maps*

$$\pi_{\alpha\beta} : G_\beta \to G_\alpha \qquad (\alpha \leq \beta < \kappa)$$

satisfying $\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma < \kappa$.
We sometimes denote such systems $\mathbf{G} = (G_\alpha, \pi_{\alpha\beta}, \kappa)$.

More concisely, such a $\mathbf{G}$ is a *contravariant functor from* κ *to* Ab; still more concisely, it is an Ab-*valued presheaf on* κ. Write $\mathsf{Pr}(\kappa, \mathsf{Ab})$ for the category of such systems (the morphisms therein are natural transformations), and write $\mathbf{\Delta}$ for the natural embedding $\mathsf{Ab} \to \mathsf{Pr}(\kappa, \mathsf{Ab})$ taking any abelian group G to the associated constant system with $G_\alpha = G$ and $\pi_{\alpha\beta} = \mathrm{id}$ for all $\alpha \leq \beta < \kappa$.

inverse limits

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\mathbf{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\mathbf{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\mathbf{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\mathbf{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

— an operation we may understand equally well in terms of the chalkboard picture(s)

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\mathbf{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\mathbf{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

— an operation we may understand equally well in terms of the chalkboard picture(s) (a definition in terms of a *universal property*)

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\boldsymbol{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\boldsymbol{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

— an operation we may understand equally well in terms of the chalkboard picture(s) (a definition in terms of a *universal property*), or the more concrete description

$$\lim \mathbf{G} = \{(g_\alpha) \in \prod_{\alpha \in \kappa} G_\alpha \mid \pi_{\alpha\beta}(g_\beta) = g_\alpha \text{ for all } \alpha \leq \beta < \kappa\}.$$

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\mathbf{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\mathbf{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

— an operation we may understand equally well in terms of the chalkboard picture(s) (a definition in terms of a *universal property*), or the more concrete description

$$\lim \mathbf{G} = \{(g_\alpha) \in \prod_{\alpha \in \kappa} G_\alpha \mid \pi_{\alpha\beta}(g_\beta) = g_\alpha \text{ for all } \alpha \leq \beta < \kappa\}.$$

Exercise

1. Verify the equivalence of these three definitions.

inverse limits

Write $\lim : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ for the reverse operation, or more precisely *right adjoint functor*, to $\mathbf{\Delta}$:

$$\mathrm{Hom}_{\mathsf{Pr}(\kappa,\mathsf{Ab})}(\mathbf{\Delta}(G), \mathbf{H}) = \mathrm{Hom}_{\mathsf{Ab}}(G, \lim \mathbf{H})$$

— an operation we may understand equally well in terms of the chalkboard picture(s) (a definition in terms of a *universal property*), or the more concrete description

$$\lim \mathbf{G} = \{(g_\alpha) \in \prod_{\alpha \in \kappa} G_\alpha \mid \pi_{\alpha\beta}(g_\beta) = g_\alpha \text{ for all } \alpha \leq \beta < \kappa\}.$$

Exercise

1. Verify the equivalence of these three definitions.

2. Construe a coherent family of functions of height κ as an element of an inverse limit indexed by κ.

the presheaf category

the presheaf category

Bear with me

the presheaf category

Bear with me: the following abstractions shed light on objects that we care about, and on their relations to further (and sometimes otherwise unsuspected) kindred objects.

the presheaf category

Bear with me: the following abstractions shed light on objects that we care about, and on their relations to further (and sometimes otherwise unsuspected) kindred objects.

Theorem

For any (small) category $\mathcal{C}$, $\mathsf{Pr}(\mathcal{C}, \mathsf{Ab})$ *is an abelian category.*

the presheaf category

Bear with me: the following abstractions shed light on objects that we care about, and on their relations to further (and sometimes otherwise unsuspected) kindred objects.

Theorem

For any (small) category $\mathcal{C}$, $\mathsf{Pr}(\mathcal{C}, \mathsf{Ab})$ *is an abelian category.*

In particular (and this is all we really want from the theorem), $\mathsf{Pr}(\kappa, \mathsf{Ab})$ is equipped for the basics of homological algebra: we can speak therein of *kernels* and *images* — (these are defined "pointwise") — and thus of *exact sequences*, short or long.

the presheaf category

Bear with me: the following abstractions shed light on objects that we care about, and on their relations to further (and sometimes otherwise unsuspected) kindred objects.

Theorem

For any (small) category $\mathcal{C}$, $\mathsf{Pr}(\mathcal{C}, \mathsf{Ab})$ *is an abelian category.*

In particular (and this is all we really want from the theorem), $\mathsf{Pr}(\kappa, \mathsf{Ab})$ is equipped for the basics of homological algebra: we can speak therein of *kernels* and *images* — (these are defined "pointwise") — and thus of *exact sequences*, short or long.

Example

Fix an abelian group A and let $G_\alpha = \bigoplus_\alpha A$ and $H_\alpha = \prod_\alpha A$ and $(H/G)_\alpha = H_\alpha / G_\alpha$ for all $\alpha < \kappa$.

the presheaf category

Bear with me: the following abstractions shed light on objects that we care about, and on their relations to further (and sometimes otherwise unsuspected) kindred objects.

Theorem

For any (small) category $\mathcal{C}$, $\mathsf{Pr}(\mathcal{C}, \mathsf{Ab})$ *is an abelian category.*

In particular (and this is all we really want from the theorem), $\mathsf{Pr}(\kappa, \mathsf{Ab})$ is equipped for the basics of homological algebra: we can speak therein of *kernels* and *images* — (these are defined "pointwise") — and thus of *exact sequences*, short or long.

Example

Fix an abelian group A and let $G_\alpha = \bigoplus_\alpha A$ and $H_\alpha = \prod_\alpha A$ and $(H/G)_\alpha = H_\alpha / G_\alpha$ for all $\alpha < \kappa$. Via the natural projection maps $\pi_{\alpha\beta}$, these assemble into inverse systems $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{H}/\mathbf{G}$ which assemble in turn into a short exact sequence in $\mathsf{Pr}(\kappa, \mathsf{Ab})$:

$$\mathbf{0} \to \mathbf{G} \to \mathbf{H} \to \mathbf{H}/\mathbf{G} \to \mathbf{0}$$

how lim got its exactness back

how lim got its exactness back

Applying the functor lim to the preceding sequence gives a short sequence of abelian groups:

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to 0$$

Is it exact?

how lim got its exactness back

Applying the functor lim to the preceding sequence gives a short sequence of abelian groups:

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to 0$$

Is it exact?

Exercise

3. Recall from Exercise 2 that A-valued coherent families of height κ represent elements of $\lim \mathbf{H}/\mathbf{G}$.

how lim got its exactness back

Applying the functor lim to the preceding sequence gives a short sequence of abelian groups:

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to 0$$

Is it exact?

Exercise

3. Recall from Exercise 2 that A-valued coherent families of height κ represent elements of $\lim \mathbf{H}/\mathbf{G}$. Conclude that the above sequence is exact if and only if every such family is trivial.

how lim got its exactness back

Applying the functor lim to the preceding sequence gives a short sequence of abelian groups:

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to 0$$

Is it exact?

Exercise

3. Recall from Exercise 2 that A-valued coherent families of height κ represent elements of $\lim \mathbf{H}/\mathbf{G}$. Conclude that the above sequence is exact if and only if every such family is trivial.

We've seen, in particular, that if $\kappa = \omega_1$ and $A = \mathbb{Z}$ then the above sequence is *not* exact (the lim functor is only *left exact*).

how lim got its exactness back

Applying the functor lim to the preceding sequence gives a short sequence of abelian groups:

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to$$

Is it exact?

Exercise

3. Recall from Exercise 2 that A-valued coherent families of height κ represent elements of $\lim \mathbf{H}/\mathbf{G}$. Conclude that the above sequence is exact if and only if every such family is trivial.

We've seen, in particular, that if $\kappa = \omega_1$ and $A = \mathbb{Z}$ then the above sequence is *not* exact (the lim functor is only *left exact*). It does, however, fit into a longer sequence which *is* exact, if we replace its rightmost 0 with a

$$\lim^1 \mathbf{G} \to \lim^1 \mathbf{H} \to \lim^1 \mathbf{H}/\mathbf{G} \to \lim^2 \mathbf{G} \to \cdots$$

how lim got its exactness back

how lim got its exactness back

This family of functors $\lim^n : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ ($n \geq 0$, with $\lim^0 = \lim$) in fact takes *any* short exact sequence to a long exact one, just as above, and this property nearly suffices to define them (they are the *universal δ-functor with initial value* $\lim$).

how lim got its exactness back

This family of functors $\lim^n : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ ($n \geq 0$, with $\lim^0 = \lim$) in fact takes *any* short exact sequence to a long exact one, just as above, and this property nearly suffices to define them (they are the *universal* δ*-functor with initial value* $\lim$). Fortunately, just like for lim, they admit a more concrete definition as well.

how lim got its exactness back

This family of functors $\lim^n : \mathsf{Pr}(\kappa, \mathsf{Ab}) \to \mathsf{Ab}$ ($n \geq 0$, with $\lim^0 = \lim$) in fact takes *any* short exact sequence to a long exact one, just as above, and this property nearly suffices to define them (they are the *universal δ-functor with initial value* lim). Fortunately, just like for lim, they admit a more concrete definition as well.

Definition

For any $\mathbf{G} \in \mathsf{Pr}(\kappa, \mathsf{Ab})$, let $\mathcal{K}(\mathbf{G})$ denote the cochain complex

$$\cdots 0 \to K^0(\mathbf{G}) \to K^1(\mathbf{G}) \to K^2(\mathbf{G}) \to \cdots$$

with $K^n(\mathbf{G}) = \prod_{\vec{\alpha} \in [\kappa]^n} G_{\alpha_0}$ and $d^n : K^n(\mathbf{G}) \to K^{n+1}(\mathbf{G})$ defined for $n \geq 0$ by

$$d^n(x)(\vec{\alpha}) = \sum_{i \leq n} (-1)^n x(\vec{\alpha}^i).$$

Define $\lim^n \mathbf{G}$ then to equal $\mathrm{H}^n(\mathcal{K}(\mathbf{G}))$.

derived limits

derived limits

As before, there's a third definition which it's a more advanced *Exercise 4* to reconcile with the previous two definitions: $\lim^n$ ($n \geq 0$) are the *derived functors* of the inverse limit functor lim.

derived limits

As before, there's a third definition which it's a more advanced *Exercise 4* to reconcile with the previous two definitions: $\lim^n$ $(n \geq 0)$ are the *derived functors* of the inverse limit functor lim. To get a handle on our second definition, let's compute the first derived limit of $\mathbf{H}$, above, whose terms are $\prod_\alpha A$ for $\alpha < \kappa$.

derived limits

As before, there's a third definition which it's a more advanced *Exercise 4* to reconcile with the previous two definitions: $\lim^n$ $(n \geq 0)$ are the *derived functors* of the inverse limit functor lim. To get a handle on our second definition, let's compute the first derived limit of $\mathbf{H}$, above, whose terms are $\prod_\alpha A$ for $\alpha < \kappa$.

Example

$\lim^1 \mathbf{H}$ equals

$$\frac{\ker(d^1 : K^1(\mathbf{H}) \to K^2(\mathbf{H}))}{\mathrm{im}(d^0 : K^0(\mathbf{H}) \to K^1(\mathbf{H}))},$$

derived limits

As before, there's a third definition which it's a more advanced *Exercise 4* to reconcile with the previous two definitions: $\lim^n$ $(n \geq 0)$ are the *derived functors* of the inverse limit functor lim. To get a handle on our second definition, let's compute the first derived limit of $\mathbf{H}$, above, whose terms are $\prod_\alpha A$ for $\alpha < \kappa$.

Example

$\lim^1 \mathbf{H}$ equals

$$\frac{\ker(d^1 : K^1(\mathbf{H}) \to K^2(\mathbf{H}))}{\operatorname{im}(d^0 : K^0(\mathbf{H}) \to K^1(\mathbf{H}))},$$

which equals

$$\frac{\{x \in K^1(\mathbf{H}) \mid \forall(\alpha,\beta,\gamma) \in [\kappa]^3 \; x(\beta,\gamma) - x(\alpha,\gamma) + x(\alpha,\beta) = 0\}}{\{x \in K^1(\mathbf{H}) \mid \exists y \in K^0(\mathbf{H}) \, \forall(\alpha,\beta) \in [\kappa]^2 \; x(\alpha,\beta) = y(\beta) - y(\alpha)\}}.$$

derived limits

As before, there's a third definition which it's a more advanced *Exercise 4* to reconcile with the previous two definitions: $\lim^n$ $(n \geq 0)$ are the *derived functors* of the inverse limit functor lim. To get a handle on our second definition, let's compute the first derived limit of $\mathbf{H}$, above, whose terms are $\prod_\alpha A$ for $\alpha < \kappa$.

Example

$\lim^1 \mathbf{H}$ equals

$$\frac{\ker(d^1 : K^1(\mathbf{H}) \to K^2(\mathbf{H}))}{\mathrm{im}(d^0 : K^0(\mathbf{H}) \to K^1(\mathbf{H}))},$$

which equals

$$\frac{\{x \in K^1(\mathbf{H}) \mid \forall(\alpha, \beta, \gamma) \in [\kappa]^3\, x(\beta, \gamma) - x(\alpha, \gamma) + x(\alpha, \beta) = 0\}}{\{x \in K^1(\mathbf{H}) \mid \exists y \in K^0(\mathbf{H})\, \forall(\alpha, \beta) \in [\kappa]^2\, x(\alpha, \beta) = y(\beta) - y(\alpha)\}}.$$

Let's try to write this better.

not a hard problem

not a hard problem

What the last quotient on the previous slide may be seen as asking is:

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$*, is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$*, is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

$\lim^1 \mathbf{H} = 0$ if and only if the answer is always *yes*.

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$, *is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

$\lim^1 \mathbf{H} = 0$ if and only if the answer is always *yes*.

Exercise

5. Show that the answer is always yes.

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$*, is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

$\lim^1 \mathbf{H} = 0$ if and only if the answer is always *yes*.

Exercise

5. Show that the answer is always yes.

One reason you'll succeed in building such a Ψ whenever you try to is because we've placed no constraints on it;

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$, *is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

$\lim^1 \mathbf{H} = 0$ if and only if the answer is always *yes.*

Exercise

5. Show that the answer is always yes.

One reason you'll succeed in building such a Ψ whenever you try to is because we've placed no constraints on it; requiring that all ψ_α and $\varphi_{\alpha\beta}$ be *finitely supported*, for example, makes for a more interesting question

not a hard problem

What the last quotient on the previous slide may be seen as asking is: *Given a family* $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma < \kappa$*, is there a* $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ *satisfying*

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha$$

for all $\alpha < \beta < \kappa$*?*

$\lim^1 \mathbf{H} = 0$ if and only if the answer is always *yes*.

Exercise

5. Show that the answer is always yes.

One reason you'll succeed in building such a Ψ whenever you try to is because we've placed no constraints on it; requiring that all ψ_α and $\varphi_{\alpha\beta}$ be *finitely supported*, for example, makes for a more interesting question — namely, the one corresponding to $\lim^1 \mathbf{G}$.

2-coherence

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$.

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$. We say that such a Φ is 2-coherent.

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$. We say that such a Φ is 2-coherent.

Such a Φ is 2-trivial if there's a $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \kappa$.

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$. We say that such a Φ is 2-coherent.

Such a Φ is 2-trivial if there's a $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \kappa$. Much as above, then:

$\lim^1 \mathbf{H}/\mathbf{G} = 0$ iff every 2-coherent Φ as above is 2-trivial.

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$. We say that such a Φ is 2-coherent.

Such a Φ is 2-trivial if there's a $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \kappa$. Much as above, then:

$\lim^1 \mathbf{H}/\mathbf{G} = 0$ iff every 2-coherent Φ as above is 2-trivial.

Where this may or may not hold is again an interesting question, and a main preoccupation of these talks.

2-coherence

Similarly, we may frame $\lim^1 \mathbf{H}/\mathbf{G}$ as a question about families $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to A \mid \alpha < \kappa \rangle$ satisfying

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} = 0 \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \gamma < \kappa$. We say that such a Φ is 2-coherent.

Such a Φ is 2-trivial if there's a $\Psi = \langle \psi_\alpha : \alpha \to A \mid \alpha < \kappa \rangle$ such that

$$\varphi_{\alpha\beta} = \psi_\beta - \psi_\alpha \qquad (\textit{mod finite})$$

for all $\alpha < \beta < \kappa$. Much as above, then:

$\lim^1 \mathbf{H}/\mathbf{G} = 0$ iff every 2-coherent Φ as above is 2-trivial.

Where this may or may not hold is again an interesting question, and a main preoccupation of these talks.
Before continuing, note the obvious

Exercise

6. Similarly characterize $\lim^n \mathbf{H}/\mathbf{G} = 0$ *for* $n > 1$.

2-coherence

2-coherence

Returning to our long exact sequence

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to \lim^1 \mathbf{H} \to \lim^1 \mathbf{H}/\mathbf{G}$$
$$\to \lim^2 \mathbf{G} \to \lim^2 \mathbf{H} \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to \lim^3 \mathbf{H} \to \cdots$$

2-coherence

Returning to our long exact sequence

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to \lim^1 \mathbf{H} \to \lim^1 \mathbf{H}/\mathbf{G}$$
$$\to \lim^2 \mathbf{G} \to \lim^2 \mathbf{H} \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to \lim^3 \mathbf{H} \to \cdots$$

which in fact cooks down to

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to 0 \to \lim^1 \mathbf{H}/\mathbf{G}$$
$$\to \lim^2 \mathbf{G} \to 0 \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to 0 \to \cdots$$

2-coherence

Returning to our long exact sequence

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to \lim^1 \mathbf{H} \to \lim^1 \mathbf{H}/\mathbf{G} \\ \to \lim^2 \mathbf{G} \to \lim^2 \mathbf{H} \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to \lim^3 \mathbf{H} \to \cdots$$

which in fact cooks down to

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to 0 \to \lim^1 \mathbf{H}/\mathbf{G} \\ \to \lim^2 \mathbf{G} \to 0 \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to 0 \to \cdots$$

we've more or less sketched the proof of the following theorem.

2-coherence

Returning to our long exact sequence

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to \lim^1 \mathbf{H} \to \lim^1 \mathbf{H}/\mathbf{G}$$
$$\to \lim^2 \mathbf{G} \to \lim^2 \mathbf{H} \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to \lim^3 \mathbf{H} \to \cdots$$

which in fact cooks down to

$$0 \to \lim \mathbf{G} \to \lim \mathbf{H} \to \lim \mathbf{H}/\mathbf{G} \to \lim^1 \mathbf{G} \to 0 \to \lim^1 \mathbf{H}/\mathbf{G}$$
$$\to \lim^2 \mathbf{G} \to 0 \to \lim^2 \mathbf{H}/\mathbf{G} \to \lim^3 \mathbf{G} \to 0 \to \cdots$$

we've more or less sketched the proof of the following theorem.

Theorem

For any $n \geq 1$, there exists a height-κ nontrivial n-coherent family of A-valued functions if and only if $\lim^n \mathbf{G} \neq 0$.

reflections

reflections

Q: Is this a hard theorem?

reflections

Q: Is this a hard theorem?

A: Not really.

reflections

Q: Is this a hard theorem?

A: Not really.

Q: What good is it?

reflections

Q: Is this a hard theorem?

A: Not really.

Q: What good is it?

A: It situates classical (1-)coherence within a wider family of combinatorial phenomena parametrized by

- domain: κ,
- codomain: A,
- dimension: n, and
- modulus: (the terms/supports defining $\mathbf{G}$).

reflections

Q: Is this a hard theorem?

A: Not really.

Q: What good is it?

A: It situates classical (1-)coherence within a wider family of combinatorial phenomena parametrized by

- domain: κ,
- codomain: A,
- dimension: n, and
- modulus: (the terms/supports defining $\mathbf{G}$).

Q: Are there any downsides to this framing?

reflections

Q: Is this a hard theorem?

A: Not really.

Q: What good is it?

A: It situates classical (1-)coherence within a wider family of combinatorial phenomena parametrized by

- domain: κ,
- codomain: A,
- dimension: n, and
- modulus: (the terms/supports defining $\mathbf{G}$).

Q: Are there any downsides to this framing?

A: Yes: (1) the community of mathematicians to whom higher derived limits mean anything is small

reflections

Q: Is this a hard theorem?

A: Not really.

Q: What good is it?

A: It situates classical (1-)coherence within a wider family of combinatorial phenomena parametrized by

- domain: κ,
- codomain: A,
- dimension: n, and
- modulus: (the terms/supports defining $\mathbf{G}$).

Q: Are there any downsides to this framing?

A: Yes: (1) the community of mathematicians to whom higher derived limits mean anything is small, and (2) the framework we've so far introduced is a bit awkward when we wish to interrelate coherence phenomena across various domains.

reflections

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes.

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk.

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk. Write $\mathcal{U}$ for the cover of κ by open proper initial segments (so that $(\mathcal{U}, \subseteq) \cong (\kappa, \leq)$).

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk. Write $\mathcal{U}$ for the cover of κ by open proper initial segments (so that $(\mathcal{U}, \subseteq) \cong (\kappa, \leq)$). Write $\mathcal{F}_A$ for the presheaf on $(\tau(\kappa), \subseteq)$ given by $U \mapsto \bigoplus_U A$ and $\mathcal{A}$ for the (pre)sheaf $U \mapsto \mathrm{Cont}(U, A)$.

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk. Write $\mathcal{U}$ for the cover of κ by open proper initial segments (so that $(\mathcal{U}, \subseteq) \cong (\kappa, \leq)$). Write $\mathcal{F}_A$ for the presheaf on $(\tau(\kappa), \subseteq)$ given by $U \mapsto \bigoplus_U A$ and $\mathcal{A}$ for the (pre)sheaf $U \mapsto \mathrm{Cont}(U, A)$. Then:

Theorem

$\lim^n \mathbf{G} \cong \mathrm{H}^n(\mathcal{U}; \mathcal{F}_A) \cong \mathrm{H}^n(\mathcal{U}; \mathcal{A}) \cong \check{\mathrm{H}}^n(\kappa; \mathcal{A}) \cong \mathrm{H}^n(\kappa; \mathcal{A})$ *for all* $A \in \mathsf{Ab}$, $n > 0$, *and ordinals* κ *of uncountable cofinality.*

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk. Write $\mathcal{U}$ for the cover of κ by open proper initial segments (so that $(\mathcal{U}, \subseteq) \cong (\kappa, \leq)$). Write $\mathcal{F}_A$ for the presheaf on $(\tau(\kappa), \subseteq)$ given by $U \mapsto \bigoplus_U A$ and $\mathcal{A}$ for the (pre)sheaf $U \mapsto \mathrm{Cont}(U, A)$. Then:

Theorem

$\lim^n \mathbf{G} \cong \mathrm{H}^n(\mathcal{U}; \mathcal{F}_A) \cong \mathrm{H}^n(\mathcal{U}; \mathcal{A}) \cong \check{\mathrm{H}}^n(\kappa; \mathcal{A}) \cong \mathrm{H}^n(\kappa; \mathcal{A})$ *for all* $A \in \mathsf{Ab}$, $n > 0$, *and ordinals* κ *of uncountable cofinality.*

The middle three terms are computed via the Čech complex, while the righmost term's sheaf cohomology.

reflections

Q: Are there remedies to these downsides? Are there other available framings of these phenomena?

A: Same question, but whatever: yes. The ones most effectively addressing both (1) and (2) above approach κ as a topoological space, as in the *second view* noted earlier in this talk. Write $\mathcal{U}$ for the cover of κ by open proper initial segments (so that $(\mathcal{U}, \subseteq) \cong (\kappa, \leq)$). Write $\mathcal{F}_A$ for the presheaf on $(\tau(\kappa), \subseteq)$ given by $U \mapsto \bigoplus_U A$ and $\mathcal{A}$ for the (pre)sheaf $U \mapsto \mathrm{Cont}(U, A)$. Then:

Theorem

$\lim^n \mathbf{G} \cong \mathrm{H}^n(\mathcal{U}; \mathcal{F}_A) \cong \mathrm{H}^n(\mathcal{U}; \mathcal{A}) \cong \check{\mathrm{H}}^n(\kappa; \mathcal{A}) \cong \mathrm{H}^n(\kappa; \mathcal{A})$ *for all* $A \in \mathsf{Ab}$, $n > 0$, *and ordinals* κ *of uncountable cofinality.*

The middle three terms are computed via the Čech complex, while the righmost term's sheaf cohomology. One value of the latter is a rich machinery for handling maps between spaces and the relations they induce between presheaves and their cohomology groups.

a last exercise

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ *as a shorthand for "there exist height-*κ *nontrivial* n*-coherent families of* A*-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.*

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ as a shorthand for "there exist height-κ nontrivial n-coherent families of A-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.

In the next two talks I'll describe two main ways to think and/or care about these groups.

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ *as a shorthand for "there exist height-*κ *nontrivial* n*-coherent families of* A*-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.*

In the next two talks I'll describe two main ways to think and/or care about these groups.

1. Tomorrow we'll see that for many n, κ, and A, the equation $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$ is independent of the ZFC axioms.

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ *as a shorthand for "there exist height-*κ *nontrivial* n*-coherent families of* A*-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.*

In the next two talks I'll describe two main ways to think and/or care about these groups.

1. Tomorrow we'll see that for many n, κ, and A, the equation $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$ is independent of the ZFC axioms. In this case, we may regard these groups' values less as invariants of a fixed topological space than as *invariants of the universe itself.*

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa; \mathcal{A}) \neq 0$ as a shorthand for "there exist height-κ nontrivial n-coherent families of A-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.

In the next two talks I'll describe two main ways to think and/or care about these groups.

1. Tomorrow we'll see that for many n, κ, and A, the equation $\mathrm{H}^n(\kappa; \mathcal{A}) = 0$ is independent of the ZFC axioms. In this case, we may regard these groups' values less as invariants of a fixed topological space than as *invariants of the universe itself.*
2. The fact, on the other hand, that $\mathrm{H}^n(\omega_n; -) : \mathsf{Ab} \to \mathsf{Ab}$ is nonzero for every $n \geq 1$ encodes *higher-dimensional* ZFC *combinatorics on ω_n generalizing some of the most fundamental combinatorics of ω_1.*

a last exercise

Exercise

7. For now, accept $\mathrm{H}^n(\kappa;\mathcal{A}) \neq 0$ *as a shorthand for "there exist height-κ nontrivial n-coherent families of A-valued functions" — i.e., as a notation which is both handy and harbors substantial further resources which any of us is free to leverage or at least read up on when the time feels right.*

In the next two talks I'll describe two main ways to think and/or care about these groups.

1. Tomorrow we'll see that for many n, κ, and A, the equation $\mathrm{H}^n(\kappa;\mathcal{A}) = 0$ is independent of the ZFC axioms. In this case, we may regard these groups' values less as invariants of a fixed topological space than as *invariants of the universe itself.*
2. The fact, on the other hand, that $\mathrm{H}^n(\omega_n; -) : \mathsf{Ab} \to \mathsf{Ab}$ is nonzero for every $n \geq 1$ encodes *higher-dimensional* ZFC *combinatorics on ω_n generalizing some of the most fundamental combinatorics of ω_1.* How do those work?

thanks

thanks

Many thanks to the organizers for the invitation

and to the audience for your attention

and for any questions which you may have.