The Complexity of Constraint Problems

III

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OUTLINE

Finite CSPs
- fun fact & summary

Finite PCSPs
- Promise CSPs
- functions > relations

  categorical ~~[illegible]~~ reality
- still better preorder

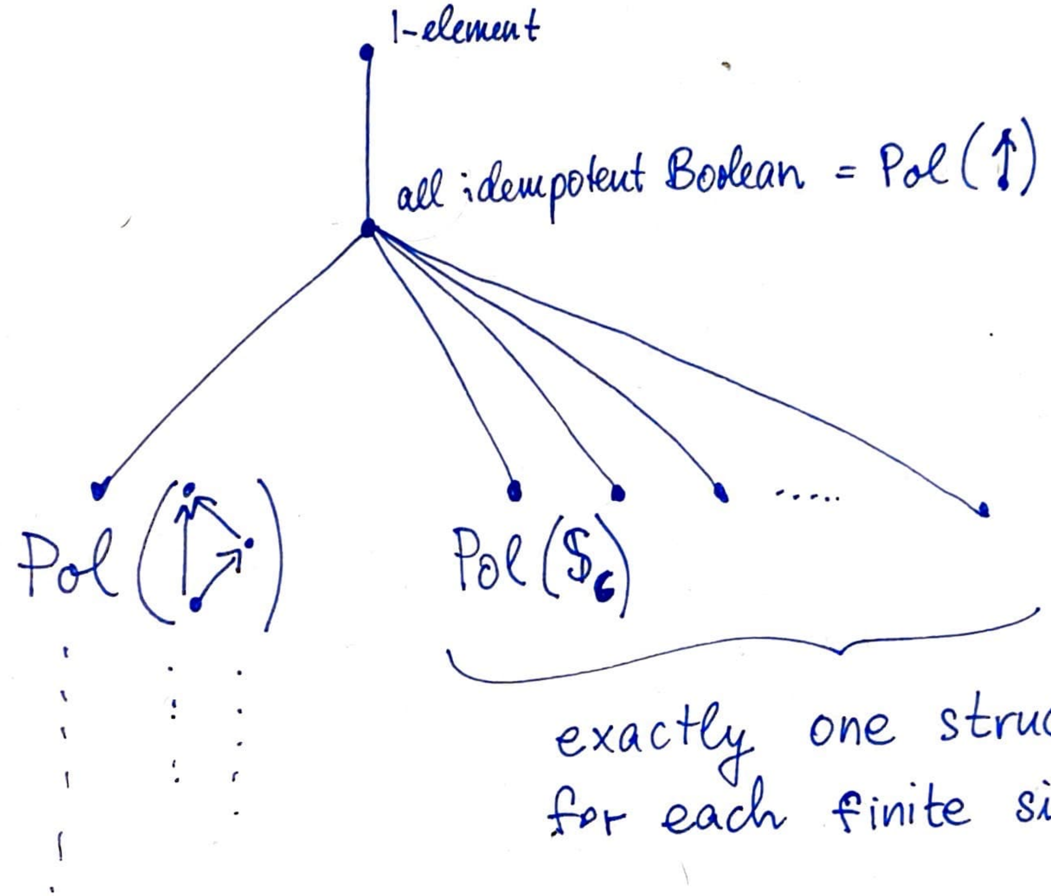
CSPs

fun fact & summary

TOP OF THE $\leq_{minion}$ ORDER

$\left(\text{"finite" clones}, \overset{minion}{\leq}\right) / \underset{\sim}{minion}$ looks like this

clones on finite sets



exactly one structure $\mathbb{S}_G$ for each finite simple group [Meyer, Starke '?]

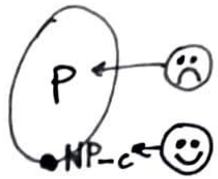
$\mathbb{S}_{A_5}$ 21-element structure found in [Carvalho, Krokhin '16]

SUMMARY

- see picture at pococop.eu

- 3 goals

uniform reduction



uniform algorithm

- $\exists$
- complex
- not very uniform (exponential in $|A|$)
- fight Gauss vs. local reasoning

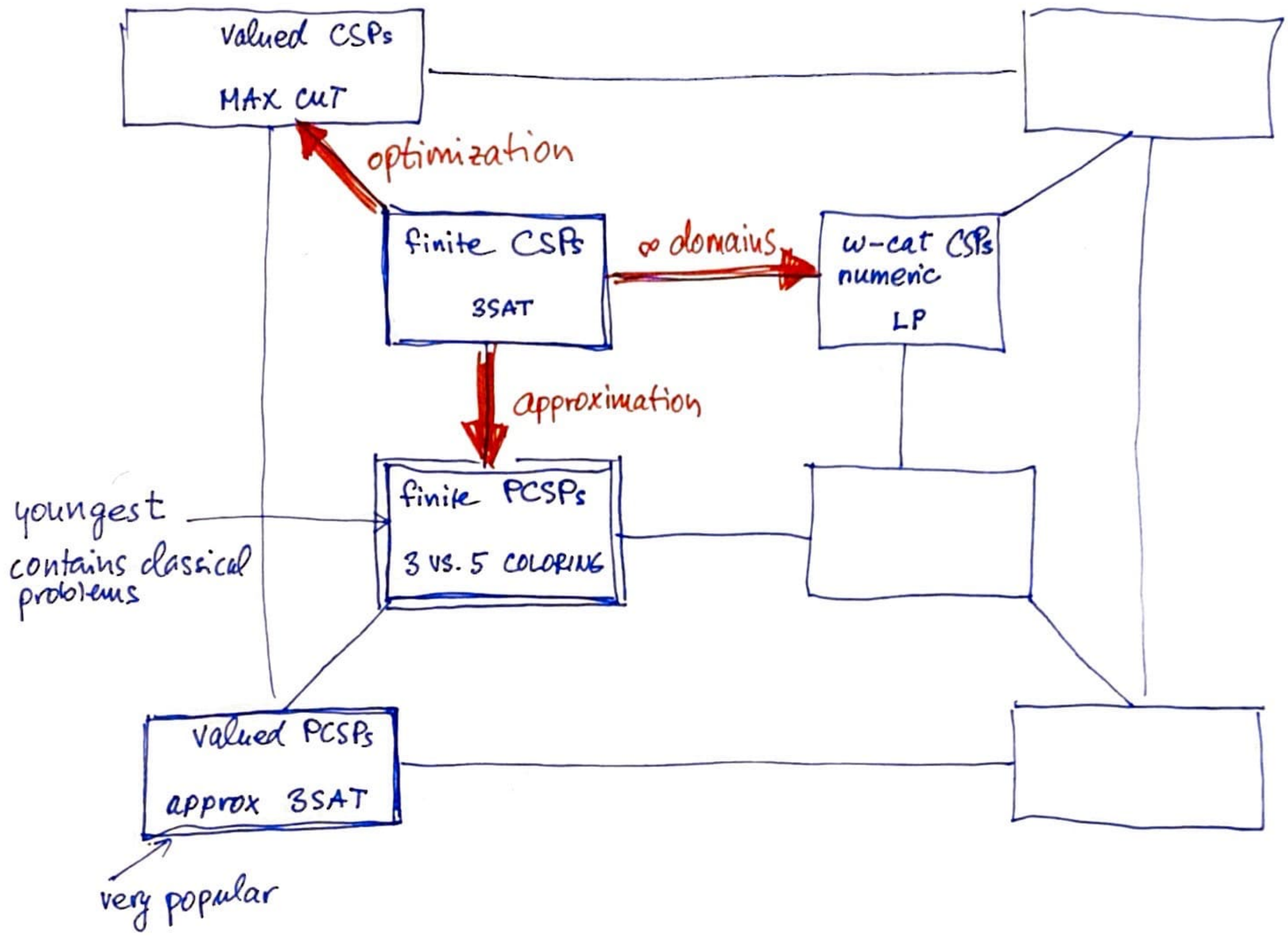
uniform description
(of borderlines)

- for P-time
- positive alternatives for $\mathcal{C} \not\to^{\text{minion}} \mathrm{Proj}$

  e.g. $c(x_1,\dots,x_n)=c(x_2,\dots,x_n,x_1)$ useful for $\mathrm{Comp}(\mathbb{A})$

Promise CSPs

THE CUBE



Valued CSPs
MAX CUT
optimization
finite CSPs
3SAT
∞ domains
ω-cat CSPs
numeric
LP
approximation
youngest
finite PCSPs
3 vs. 5 COLORING
contains classical problems
Valued PCSPs
approx 3SAT
very popular

PROMISE CSP

Fix $\mathbb{A}, \mathbb{B}$ such that $\mathbb{A} \to \mathbb{B}$

PCSP($\mathbb{A}, \mathbb{B}$) (search version)

INPUT: finite $\mathbb{X}$ s.t. $\exists \mathbb{X} \to \mathbb{A}$ → the promise

OUTPUT: $\mathbb{X} \to \mathbb{B}$

$\boxed{\text{Comp}(\mathbb{A}, \mathbb{B}) \text{ also makes sense!}}$

Examples

- PCSP($\mathbb{K}_3, \mathbb{K}_5$) Find a 5-coloring of a given 3-colorable graph
- PCSP($\mathbb{H}_2, \mathbb{H}_7$) Find a 7-coloring of a given 2-colorable 3-uniform hypergraph

  $\mathbb{H}_2$: ternary not-all-equal relation on $\{0,1\}$
- PCSP(1-in-3, $\mathbb{H}_2$) Boolean PCSP

  1-in-3: $\{(0,0,1),(0,1,0),(1,0,0)\}$

  1-in-3-SAT vs. NAE-SAT

CSP($\mathbb{K}_3$) = PCSP($\mathbb{K}_3, \mathbb{K}_3$) NP-complete

PCSP($\mathbb{K}_3, \mathbb{K}_4$) NP-complete [Khanna, Linial, Safra '00]

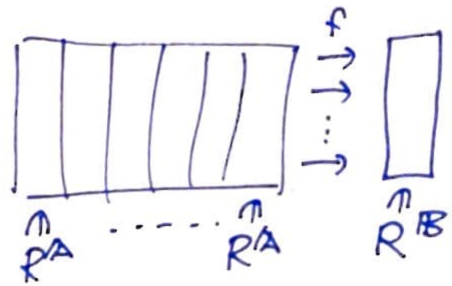
PCSP($\mathbb{K}_3, \mathbb{K}_5$) NP-complete [Bulín, Krokhin, Opršal '19]

PCSP($\mathbb{K}_3, \mathbb{K}_6$) ?

PCSP($\mathbb{H}_k, \mathbb{H}_\ell$) NP-complete, $k \leq \ell$ [Dinur, Regev, Smyth '05]

in P [Brakensiek, Guruswami '18]

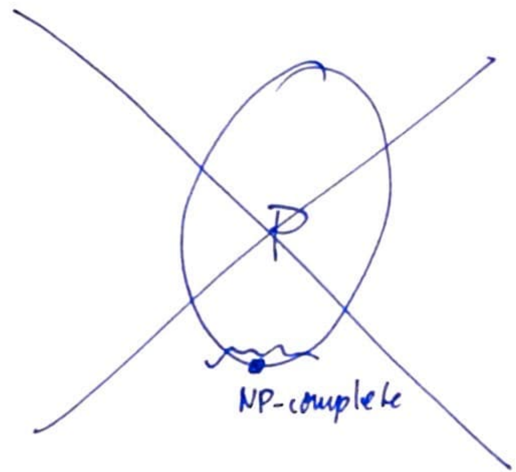
BASICS WORK, BUT...



- polymorphism of $(\mathbb{A}, \mathbb{B})$ $\quad f: A^n \longrightarrow B$ s.t.
- $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ = all polymorphisms
    - not a clone
    - closed under taking minors ... <u>minion</u> [B, Bulín, Krokhin, Opršal '21]
    - captures complexity $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \overset{\text{minion}}{\leq} \mathrm{Pol}(\mathbb{C}, \mathbb{D}) \Rightarrow \mathrm{PCSP}(\mathbb{C}, \mathbb{D}) \leq \mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

    applications
    - $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_4) \overset{\text{minion}}{\leq} \mathrm{Proj} \Rightarrow \mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ NP-complete
    - $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5) \overset{\text{minion}}{\leq} \mathrm{Pol}(\mathbb{H}_2, \mathbb{H}_{100000}) \Rightarrow$ also NP-complete

- hardness more interesting
    - $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5) \overset{\text{minion}}{\not\leq} \mathrm{Proj}$ but still NP-hard
    - $\mathrm{Pol}(\mathbb{H}_2, \mathbb{H}_7) \not\leq \mathrm{Proj}$



ALGORITMICALLY ALSO INTERESTING

- $CSP(\mathbb{A})$ in P or $CSP(\mathbb{B})$ in P $\Rightarrow$ $PCSP(\mathbb{A}, \mathbb{B})$ in P
- But $CSP(\text{1-in-3})$ NP-complete
  $CSP(\mathbb{H}_2)$ NP-complete
  $PCSP(\text{1-in-3}, \mathbb{H}_2)$ in P [board]
- In fact $\text{1-in-3} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{H}_2$

  integers relation $x+y+z=1$

  and $CSP(\mathbb{Z})$ in P [Frumkin'76, Kannan, Bachem '79]
- In fact if $\text{1-in-3} \longrightarrow \mathbb{C} \longrightarrow \mathbb{H}_2$ and $CSP(\mathbb{C})$ in P $\Rightarrow$ $\mathbb{C}$ infinite [B'19]

functions
> relations

DIAGRAMS

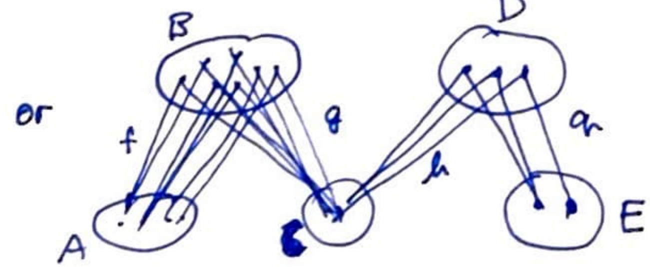
- <u>shape</u>   finite category
  (∘, id typically not important here)

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet & & \bullet \end{tikzcd}$$

- <u>diagram of shape S</u>
  functor $S \to \mathcal{F}in\mathcal{S}et_{\neq\emptyset}$
  finite nonempty sets

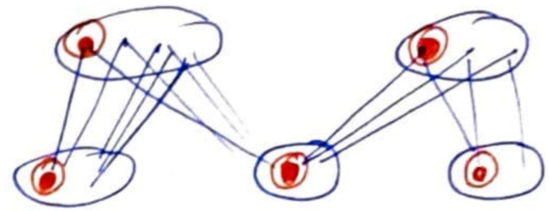
$$\begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "g"] & & D \arrow[dl, "h"'] \arrow[dr, "q"] & \\ A & & C & & E \end{tikzcd}$$

or



- <u>solution of diagram $\mathcal{D}$</u>
  element of $\lim \mathcal{D}$

$a \in A,\ b \in B, \dots$
s.t. $f(b) = a,\ g(b) = c, \dots$



- CSP instance $\mapsto$ diagram
  say $\mathbb{A} = (A;\ R \subseteq A^2)$

$\exists x,y,z \quad R(x,y) \wedge R(y,z) \quad \longmapsto$

$$\begin{tikzcd} & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & A & & A \end{tikzcd}$$

(in particular, solving diagrams is NP-complete)

SOLVING DIAGRAMS IS EVERYWHERE...

- PCP THEOREM + PARALLEL REPETITION [Arora, Safra '98, Arora, Lund, Motwani, Sudan, Szegedy '98] [Raz '98]

  INPUT: solvable diagram

  OUTPUT: selection consistent with at least 0.0001% functions

- KÖNIG's TREE LEMMA: [König '27]

  every diagram of shape $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \cdots$ has a solution

- Ramsey property of a class of structures can be phrased [Hadek '25]

  every diagram of shape ..... has a solution

... INCLUDING (P)CSP

- <u>abstract minion</u> functor $\mathrm{FinSet}_{\neq\emptyset} \to \mathrm{FinSet}_{\neq\emptyset}$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}(X) \arrow[d, "\mathcal{M}(\pi)"] \\ \mathcal{M}(Y) \end{tikzcd}$$

- minion $\longmapsto$ abstract minion

$$\mathcal{M} \qquad \begin{tikzcd} X \arrow[d, "\pi"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} X\text{-ary members of } \mathcal{M} \arrow[d, "\text{take minor via } \pi"] \\ Y\text{-ary members of } \mathcal{M} \end{tikzcd}$$

THEOREM [B, Bulín, Krokhin, Opršal '21] *slightly lying

$\mathbb{A} \to \mathbb{B}$ finite. $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ is equivalent to

$\mathcal{M} = \mathrm{Pol}(\mathbb{A},\mathbb{B})$ $\quad \mathrm{MC}(\mathcal{M})$ $\quad$ INPUT: diagram $\mathcal{D}$ having a solution

OUTPUT: solution of $\mathcal{M}\mathcal{D}$

$$\begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & C \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{M}(B) \arrow[dl, "\mathcal{M}(f)"'] \arrow[dr, "\mathcal{M}(g)"] & \\ \mathcal{M}(A) & & \mathcal{M}(C) \end{tikzcd}$$

Example

- $\mathrm{Pol}(\text{Horn-SAT})$ is

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \text{nonempty subsets of } X \arrow[d, "\pi\text{-image}"] \\ \text{— '' — of } Y \end{tikzcd}$$

- solve $\mathrm{MC}(\mathcal{M})$ for $\mathcal{M} = \mathrm{Pol}(\text{Horn-SAT})$ [board]

NICE MINION

- $\mathcal{M} = \mathrm{Clo}\,(\, xy^{-1}z$ in dihedral group $D_8 = \mathrm{Aut}(\square)\,)$
  $= \mathrm{Pol}\,($ some 8-element structure $\mathbb{A}\,)$

3+ algorithms for $\mathrm{CSP}(\mathbb{A})$

- group-like problem in Feder-Vardi '98
- describing all solutions-like algorithm in Bulatov, Dalmau '06
- Zhuk's CSP algorithm '17

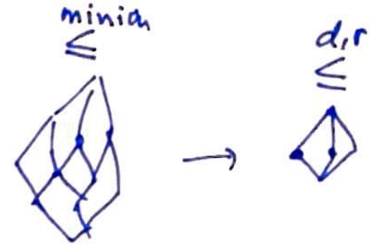
new algorithm for $\mathrm{MC}(\mathcal{M})$

☺ no fight between Gauss & local reasoning

Still better preorder

D,R-HOMOMORPHISMS

even weaker than minion homo



THEOREM [B, Kozik '22]

If $Pol(\mathbb{A},\mathbb{B}) \overset{d\text{-}r}{\leq} Pol(\mathbb{C},\mathbb{D})$ then $PCSP(\mathbb{C},\mathbb{D}) \leq PCSP(\mathbb{A},\mathbb{B})$

THM? [Barto]

$Pol(\mathbb{A},\mathbb{B}) \overset{d,r}{\leq} Proj \Rightarrow$

$Comp(\mathbb{A},\mathbb{B})$ implies ultrafilter axiom

~~∃ minion homo~~

$\mathcal{M} \overset{d,r}{\leq} Proj$ if (not only if)

∃ function I from $\mathcal{M}$ s.t. $\exists d \in \mathbb{N}$

- $\forall f \in \mathcal{M}\ f: A^n \to B \quad I(f) \subseteq \{1,2,\dots,n\} \quad |I(f)| \leq d$
- $\forall f \in \mathcal{M}\ f: A^n \to B\ \forall \pi \quad I(f^{(\pi)}) \cap \pi(I(f)) \neq \emptyset$

common feature: extensively use algebraic topology

Examples: $Pol(\mathbb{H}_2, \mathbb{H}_7) \overset{d,r}{\leq} Proj$, $Pol(\mathbb{C}_{137}, \mathbb{K}_3) \overset{d,r}{\leq} Proj$, $Pol(\mathbb{C}_{137}, \mathbb{K}_4) \overset{d,r}{\leq} Proj$

[board]

[Dinur, Regev, Smythe '03]
[B, Bulín, Krokhin, Opršal '21]

[Krokhin, Opršal '19]

[Avakumov, Filakovský, Opršal, Tasinato, Wagner '25]

DECODING MULTIASSIGNMENTS

- Fix m (e.g. $m=2$), $k > \ell$, d (e.g. $d=2$)
- Given
  - a CSP instance with all constraints of arity $\leq m$ (possibly infinite domain!)
  - for each k-element set of variables U a set $S(U)$ of partial solutions (for U) s.t. $|S(U)| \leq d$
  - dtto for each ℓ-element set of variables V
  - $\forall U \supseteq V$, $|U|=k$, $|V|=\ell$ some element of $S(U)$ is consistent with some element of $S(V)$

Does the CSP instance have a solution?

- Combinatorial core of the previous theorem: $\forall m, d\ \exists k, \ell$ the answer is YES

Wrap up

Takeaways

- interaction CSP $\leftrightarrow$ set theory
- clones & minions : useful generalizations of permutation groups
- abstraction useful   perm. group $\to$ group
  clone/minion $\to$ abstract minion
- absorption useful
- (P)CSP & other stuff = solving diagrams

Reading

fin. CSP   Barto, Krokhin, Willard: Polymorphisms & how to use them

Barto, Kozik: Absorption in Universal Algebra and CSP

fin. PCSP   Barto, Bulín, Krokhin, Opršal: Algebraic approach to promise constraint satisfaction

Krokhin, Opršal: An invitation to the promise constraint satisfaction problem

∞ CSP   Bodirsky: Complexity of Infinite-Domain Constraint Satisfacion

Thank you!