The Complexity of Constraint Problems

II

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RECAP & OUTLINE

clone on A = set of functions $A^n \to A$ $(n=1,2,3,\ldots)$ closed under composition

CSP($\mathbb{A}$)

INPUT: finite $\mathbb{X}$

OUTPUT: $\mathbb{X} \to \mathbb{A}$?

$\exists, \wedge, =$

INPUT: pp-sentence over $\mathbb{A}$

OUTPUT: true?

also use CSP(Γ)

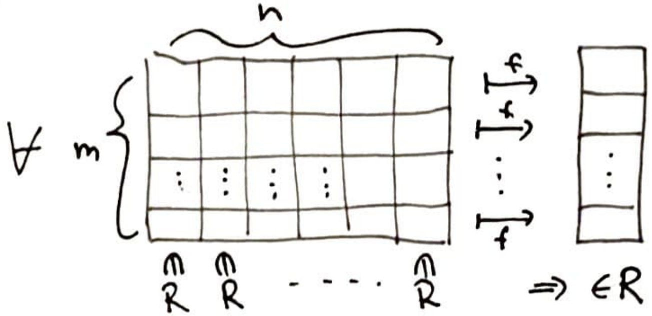
today

- clones vs. CSPs – finishing the story
- $\mathbb{K}_3$ is very asymmetric
- Comp($\mathbb{K}_3$) $\Rightarrow$ ultrafilter axiom – finishing the proof
- absorption

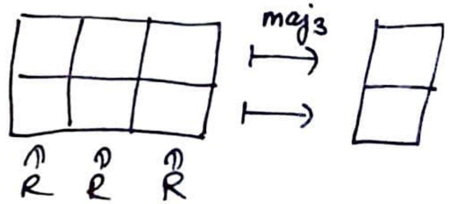
clones vs. CSPs

I. CLONES $\leftrightarrow$ CSPs : POLYMORPHISMS

- $f : \{0,1\}^n \to \{0,1\}$ is a polymorphism of $R \subseteq \{0,1\}^m$ if



maj_3 is a polymorphism of every $R \subseteq \{0,1\}^2$



- Γ ... set of Boolean relations

Pol(Γ) ... polymorphisms of every $R \in \Gamma$. $\boxed{\text{Clone}}$

$\mathrm{Pol}(\text{all binary}) = \mathrm{Clo}(\mathrm{maj}_3)$

- THEOREM [Geiger'68, Bodnarchuk, Kaluzhnin, Kotov, Romov'69]

Γ pp-definable from $\Delta \iff \mathrm{Pol}(\Gamma) \supseteq \mathrm{Pol}(\Delta)$

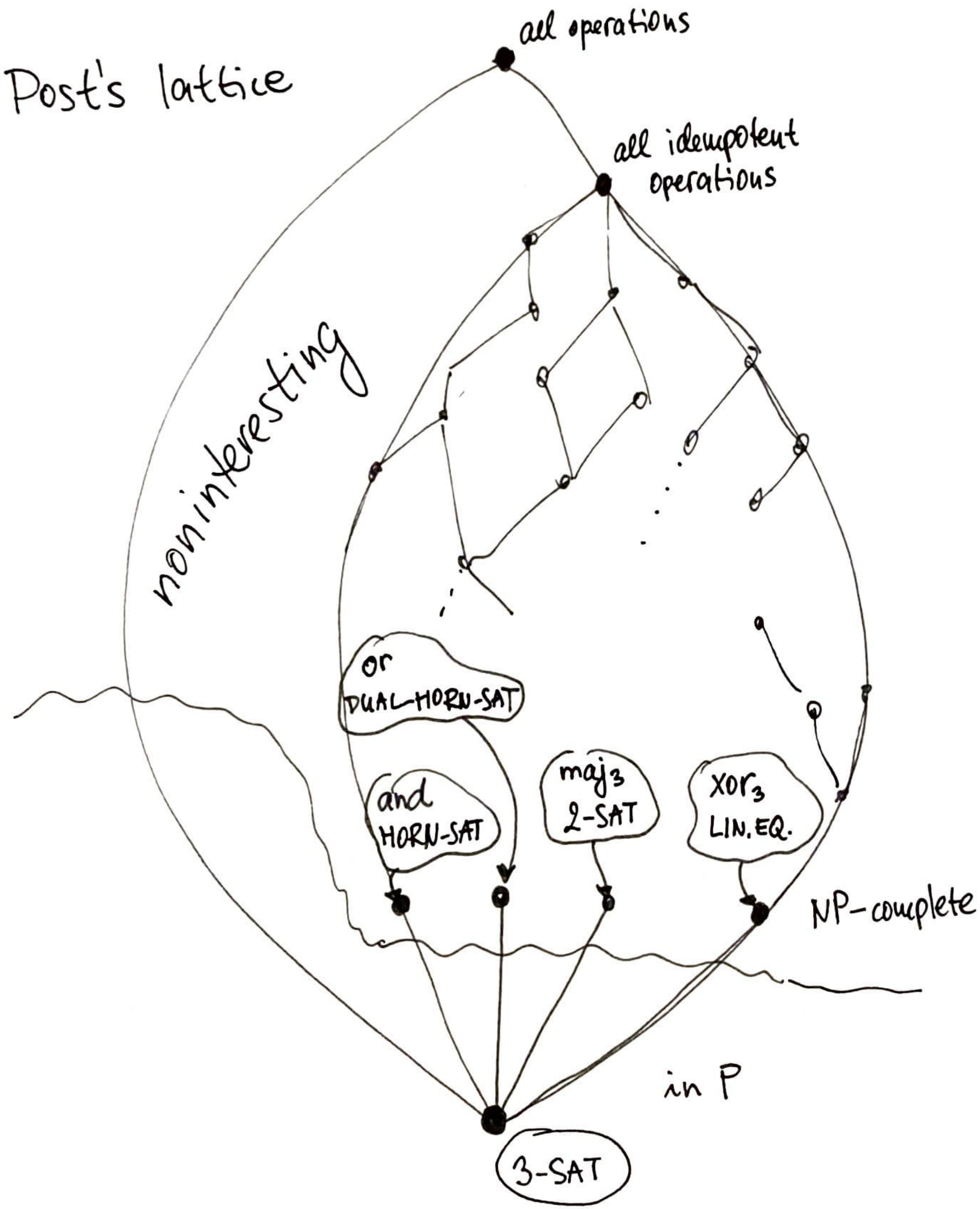
(and then $\mathrm{CSP}(\Gamma) \leq \mathrm{CSP}(\Delta)$)

[Schaefer]

not only Boolean
... any finite A

I. CLONES ⟷ CSPs: SCHAEFER REVISITED

Post's lattice



- 60s theorem implies

  $\mathrm{Pol}(\Gamma)$ higher

  $\Rightarrow$ $\mathrm{CSP}(\Gamma)$ easier

- restrict to "interesting clones"

  = idempotent, i.e. $\forall a \ f(a,a,\dots,a) = a$

- 4 minimal clones by Post
  - $\mathrm{Clo}(\mathrm{and})$
  - $\mathrm{Clo}(\mathrm{or})$
  - $\mathrm{Clo}(maj_3)$
  - $\mathrm{Clo}(xor_3)$

- they correspond to CSPs in P

  $\Rightarrow$ Schaefer

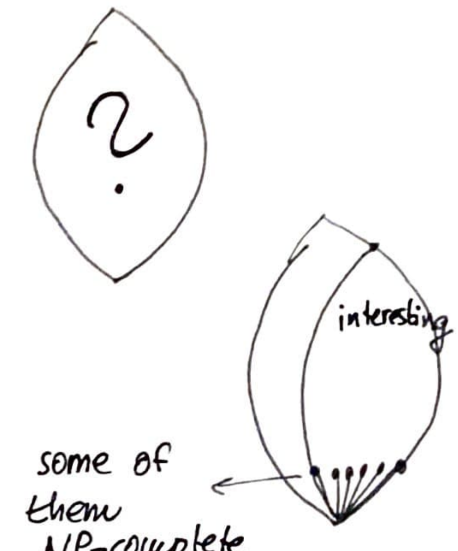
I. NEW ORDERS

- 2 problems beyond Boolean

  (1) classification hard even for clones on $\{0,1,2\}$

  (2) for CSPs: borderline is not at the bottom

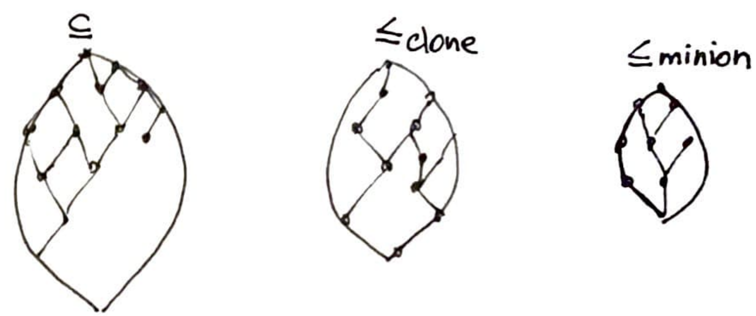
clones on $\{0,1,2\}$



- "better preorders" $\mathcal{C} \subseteq \mathcal{D} \Rightarrow \mathcal{C} \leq_{clone} \mathcal{D} \Rightarrow \mathcal{C} \leq_{minion} \mathcal{D}$

  ⇝ "simpler" posets

  still good for CSPs



  ⇝ "simpler" objects than clones

  (here "simpler" = carries less information)

I. NEW ORDERS: DETAILS

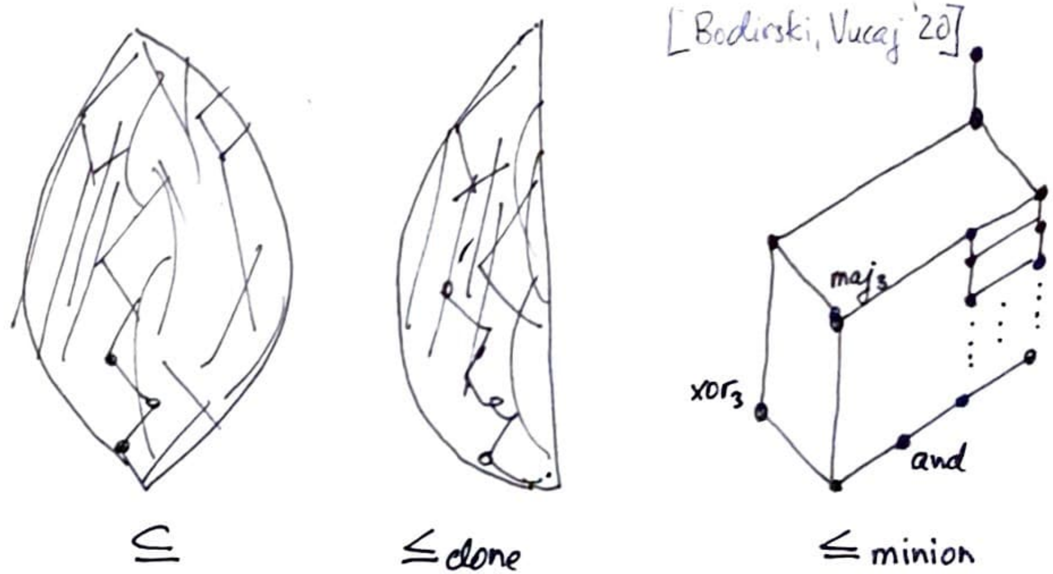
(my life ① minions ② algebras, clones, CSPs ③ minions)

$CSP(\Gamma) \leq CSP(\Delta)$ if

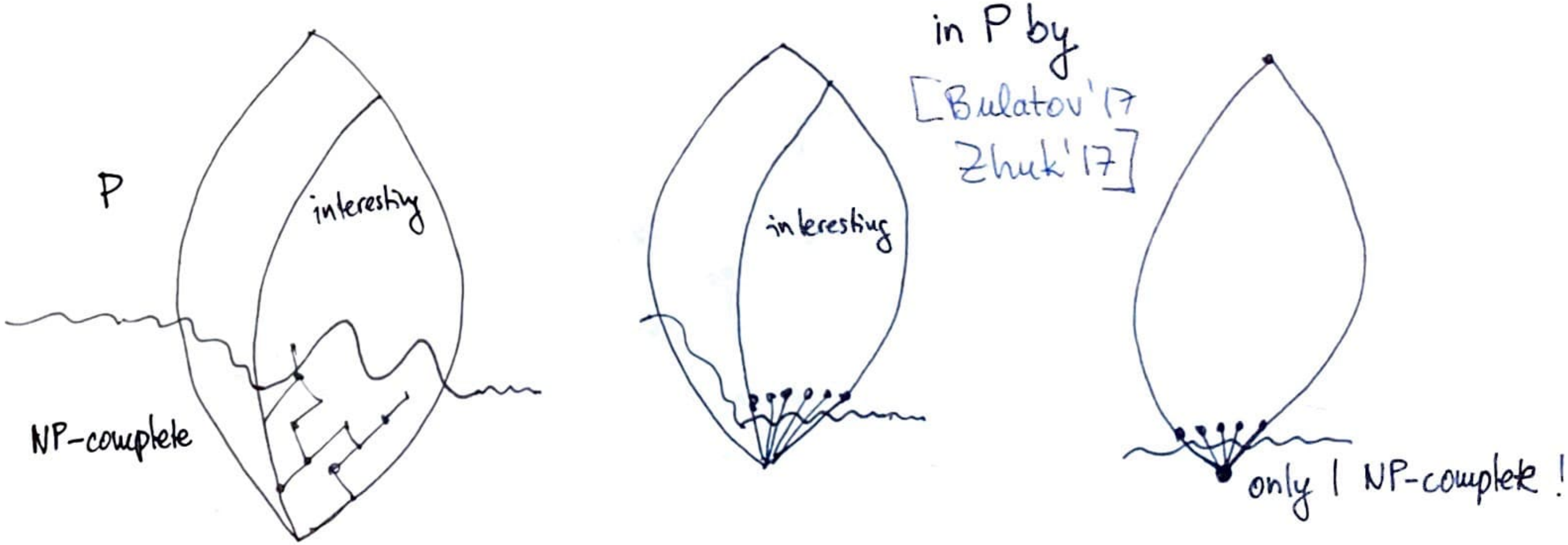
relationally	Γ pp-definable from Δ	Γ pp-interpretable in Δ	Γ pp-constructible from Δ
algebraically	$Pol(\Delta) \subseteq Pol(\Gamma)$	$\exists\, Pol(\Delta) \to Pol(\Gamma)$ clone homomorphism preserves arities & composition	$\exists\, Pol(\Delta) \to Pol(\Gamma)$ minion homomorphism preserves arities & permutation, identification of variables
preorder on clones	$\mathscr{C} \subseteq \mathscr{D}$	$\mathscr{C} \leq_{clone} \mathscr{D}$ if $\exists$ clone homo $\mathscr{C} \to \mathscr{D}$	$\mathscr{C} \leq_{minion} \mathscr{D}$ if $\exists$ minion homo $\mathscr{C} \to \mathscr{D}$
algebraic object	clone [Geiger'68; Bodnarchuk, Kaluzhnin, Kotov, Romov'69 Jeavons'00]	abstract clone [Birkhoff'35 Bulatov, Jeavons, Krokhin'05 Bodirsky'08]	minion functor finite ordinals $\to$ sets [B, Opršal, Pinsker'18 B, Bulín, Krokhin, Opršal'21] * see my life

I. NEW ORDERS: GAINS

- Boolean clones <u>get</u> simpler



- new hope for $\{0,1,2\}$ (©Zhuk) or even bigger, e.g. possibly countable!
- beautiful borderlines for CSPs



DECIPHERING "$\mathscr{C}$ IS AT THE BOTTOM"

DEF $\quad f : A^n \to A, \quad \pi : \{1,2,\dots,n\} \to \{1,2,\dots,m\}$

<u>Minor of f via π</u> is $f^{(\pi)} : A^m \to A \qquad f^{(\pi)}(a_1,\dots,a_m) = f(a_{\pi(1)},\dots,a_{\pi(n)})$

EXAMPLE $\quad$ If $g(a,b) = f(a,a,b)$ then $g = f^{(\pi)}$ for $\pi : 1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 2$

$\xi : \mathscr{C} \to \mathscr{D}$ is a minion homomorphism iff it preserves arities and minors

$\mathscr{C}$ is at the bottom

$\iff$ $\exists$ minion homo $\mathscr{C} \to \mathrm{Proj}$ $\qquad (\iff \mathscr{C} \overset{\text{minion}}{\leq} \mathrm{Proj})$

$\iff$ $\exists$ function I from $\mathscr{C}$ s.t. $\qquad$ "I selects an important coordinate"

- $\forall f \in \mathscr{C}$ $\ f : A^n \to A \qquad I(f) \in \{1,2,\dots,n\}$
- $\forall f \in \mathscr{C}$ $\ f : A^n \to A$ $\ \forall \pi \qquad I(f^{(\pi)}) = \pi(I(f))$

EXAMPLES $\quad$ [board]

ALGORITHMS

- 2 ideas
  - local consistency
    constraint propagation
    bounded width
    least fixed point logic — e.g. HORN-SAT, 2-SAT
  - describing all solutions
    (as in Gaussian elimanation) — eg. $\mathbb{Z}_p$-LIN

- Applicability understood [Ba, Kozik', Berman, Idziak, Marković, McKenzie, Valeriote, Willard']

- Bulatov / Zhuk algorithms – nontrivial combination of these
  (+ recursion, ...)

DICHOTOMY

THEOREM $\mathbb{A}$ finite, $\mathcal{C} = \mathrm{Pol}(\mathbb{A})$. One of the following happens.

- $\mathcal{C} \overset{\text{minion}}{\leq} \mathrm{Proj}$

  $\iff$ $\mathbb{A}$ pp-constructs every finite $\mathbb{B}$

  $\implies$ $\forall \mathbb{B}$ finite $\mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$

  $\implies$ –"– $\mathrm{Comp}(\mathbb{B}) \leq \mathrm{Comp}(\mathbb{A})$

  $\implies$ $\mathrm{CSP}(\mathbb{A})$ NP-complete

  $\implies$ $\mathrm{Comp}(\mathbb{A}) \implies$ ultrafilter axiom

abstract nonsense

+ $\mathrm{CSP}(\mathbb{K}_3)$ hard
+ $\mathrm{Comp}(\mathbb{K}_3)$ strong

- $\mathcal{C} \overset{\text{minion}}{\not\leq} \mathrm{Proj}$

  [Kearnes, Marković, McKenzie '14]

  $\iff$ $\mathcal{C}$ has 4-ary s s.t. $s(a,r,e,a) = s(r,a,r,e)$

  $\iff$ $\mathcal{C}$ has n-ary c s.t. $c(a_1, \dots, a_n) = c(a_2, \dots, a_n, a_1)$ ($n \geq 2$)

  ~~Barto~~ [B, Kozik '12]

  $\implies$ $\mathrm{CSP}(\mathbb{A})$ in P

  $\implies$ $\mathrm{Comp}(\mathbb{A}) \not\implies$ ultrafilter axiom

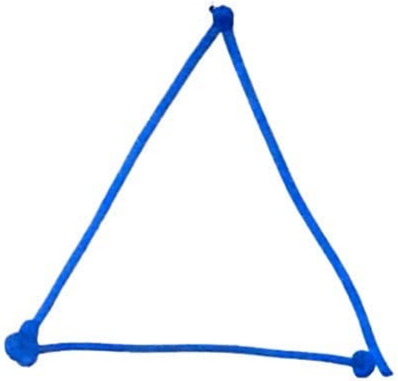
tame congruence theory

commutator theory

absorption

+ algorithm
+ constructing models

$\mathbb{K}_3$ is asymmetric



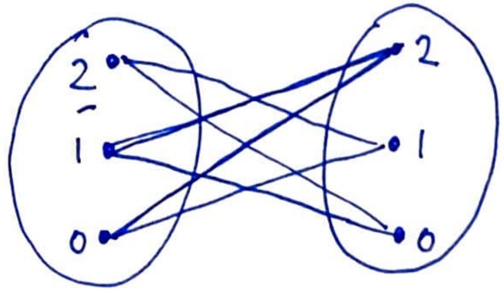
$\mathrm{Pol}(\triangle)$

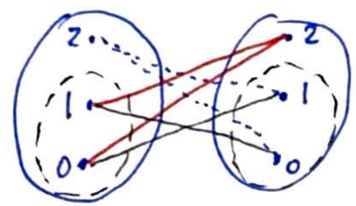
Claim: $f: \{0,1,2\}^n \to \{0,1,2\}$ in $\mathrm{Pol}(\mathbb{K}_3)$ iff $\exists i\ \exists \sigma$ permutation $\forall \bar{x}\ f(\bar{x}) = \sigma(x_i)$

Enough: f idempotent ($f(aa\ldots a) = a$) then $\exists i\ \forall \bar{x}\ f(\bar{x}) = x_i$

- $n = 2$ [board]
- Case I: $\forall x, y\ f(y, x, x, \ldots, x) = y$

  then $f(x_1, \ldots, x_n) = x_1$ $(\forall \bar{x})$ [board]
- Case II: $\forall x, y\ f(y, x, x, \ldots x) = f(x, y, x, x, \ldots, x) = \ldots = x$
  - $f(x, 2, 2, \ldots, 2) = 2$ implies $f(x, 0/1, 0/1, \ldots, 0/1) = 0/1$
  - draw $\mathbb{K}_3$ as a bipartite graph, absorb the connection from 0 to 1 into $\{0,1\}$ ↯





[board]

$$\mathrm{Comp}(\mathbb{K}_3)$$

$$\Downarrow$$

ultrafilter axiom

$\boxed{\text{COMP}(\mathbb{K}_3) \Rightarrow \text{ULTRAFILTER AXIOM}}$

- $\mathcal{F}$ filter on X. Want $\mathcal{F} \subseteq \mathcal{U}$ ultrafilter
- graph $\mathbb{G}$ vertex set $\{0,1,2\}^X$

  $(a_x)_{x \in X}$ — $(b_x)_{x\in X}$ iff $\{x;\ a_x \neq b_x\} \in \mathcal{F}$

😮 $\forall$finite $\mathbb{G}' \subseteq \mathbb{G}$ is 3-colorable

$\Rightarrow \exists c : \{0,1,2\}^X \longrightarrow \{0,1,2\}$ 3-coloring of $\mathbb{G}$ (For $\mathcal{F} = \{X\}$ = X-ary polymorphism of $\mathbb{K}_3$)

😮 WLOG $c(00\ldots0) = 0$, $c(11\ldots1) = 1$, $c(22\ldots2) = 2$

- $U \in \mathcal{U}$ iff $c(000\ldots\underbrace{11111}_{U}00\ldots0) = 1$

  - $\mathcal{U}$ extends $\mathcal{F}$
  - $\cap, \subseteq$
  - ultra

Absorption

give up your selfishness, and you shall find peace
like water mingling with water, you shall merge in **absorption**

Sri Guru Granth Sahib

ABSORPTION IS COOL

DEF B <u>absorbs</u> A wrt. $f: A^n \to A$ if $\forall i \; f(B, B, \dots, B, \overset{i}{A}, B, B, \dots, B) \subseteq B$

EX $\{1\}$ absorbs $\{0,1\}$ wrt. max, $\{0\}, \{1\}$ absorb $\{0,1\}$ wrt. maj_3

- absorption is common

[B, Kozik '12]

THM "absorption thm"

If
- $R \subsetneq A^2$ linked (connected as a bipartite graph)
- $\mathrm{IdPol}(R) \overset{\text{minion}}{\not\to} \mathrm{Proj}$

Then $\exists \emptyset \neq B \subsetneq A \; \exists f \in \mathrm{IdPol}(R)$ B absorbs A wrt. f

- absorbs stuff

[B, Kozik '09]

THM "loop lemma"

If
- $R \subseteq A^2$ linked
- $\mathrm{Pol}(R) \overset{\text{minion}}{\not\to} \mathrm{Proj}$

Then $\exists a \; (a,a) \in R$

$\Rightarrow$ e.g. dichotomy for CSP(graph) [Hell, Nešetřil '90], ...

SIGGERS OPERATION

[Siggers '10]

THM TFAE for a clone $\mathcal{C}$ on finite A

- $\mathcal{C} \overset{\text{minion}}{\not\leq} \text{Proj}$
- $\exists s \in \mathcal{C}$ 6-ary $\quad s(x,y,x,z,y,z) = s(y,x,z,x,z,y) \quad (\forall x,y,z)$

Proof of $\Downarrow$

- F = 3-ary members of $\mathcal{C}$ $\subseteq A^{A^3}$
- $R \subseteq F^2 \quad \begin{pmatrix} s(x,y,x,z,y,z) \\ s(y,x,z,x,z,y) \end{pmatrix}$ for 6-ary s

  - R linked
  - $\text{Pol}(R) \not\leq \text{Proj}$ (as $\mathcal{C} \overset{\text{minion}}{\leq} \text{Pol}(R)$)

  $\Rightarrow$ a loop