The Complexity of Constraint Problems

I

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OUTLINE OF PART 1

CSP = Constraint Satisfaction Problem

Find values for variables so that given constraints are satisfied

Question: What is the $\boxed{\text{computational complexity}}$?

- Ideal world of computational complexity
- Why am I looking at CSP (despite disliking CS)?
- Problems CSP($\mathbb{A}$)
- Why am I here?
- Story about logic & CSP

Ideal world of computational complexity

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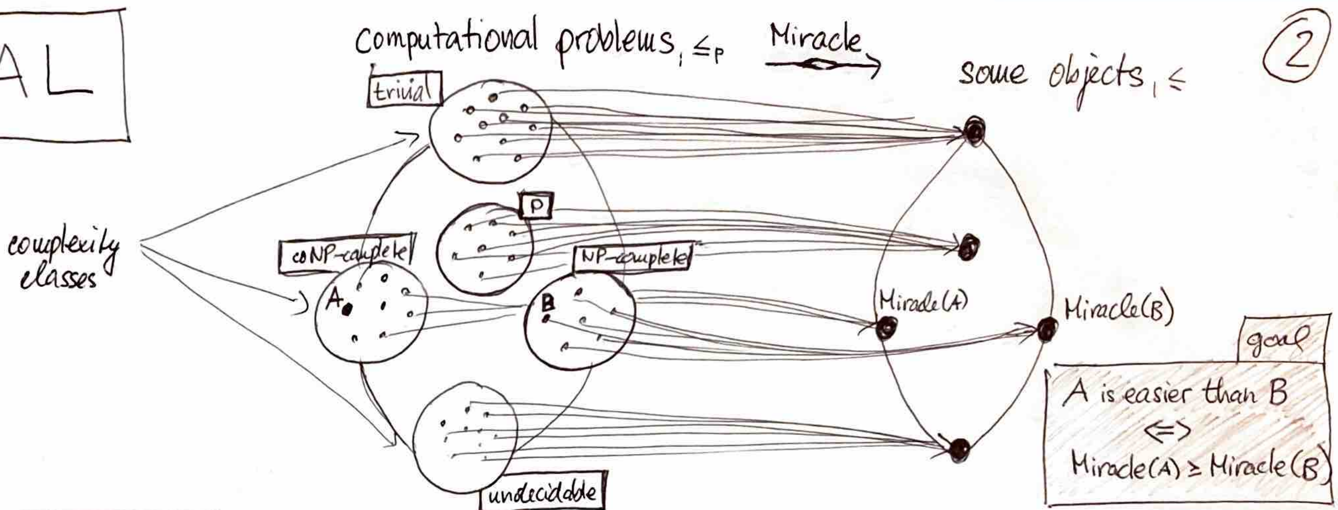
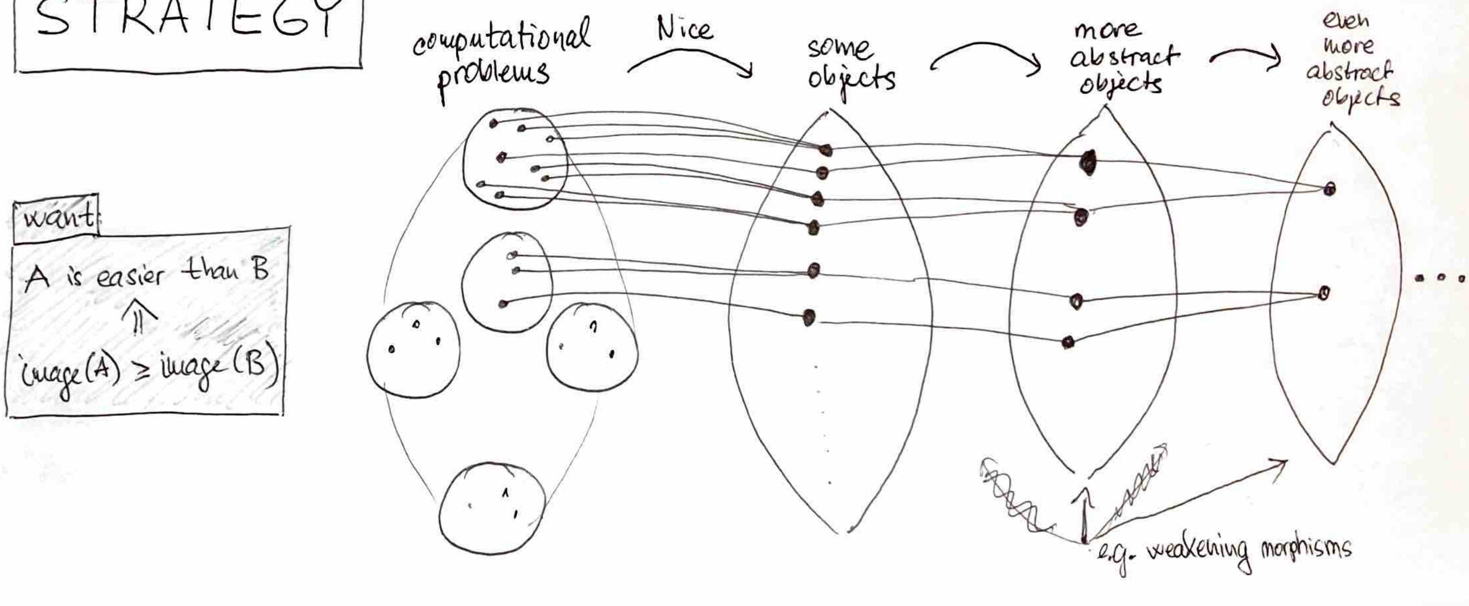
GOAL



STRATEGY



- Problems
- Why am I here?
- Story about logic & CSP

Why am I looking at CSPs?

I'M NAÏVE

my choice

little me

life choice

- have a tiny chance of starting a **new** scientific revolution ✓
- have a good chance of proving cool results

- computational complexity (even basic questions like $P = NP$?) seems to require substantially new math

- very popular: composing unary functions (permutation groups, groups, monoids, ...)
  not so popular: — " — multivariable functions (**clones**, minions)

- math is too linear-algebraic

CSP

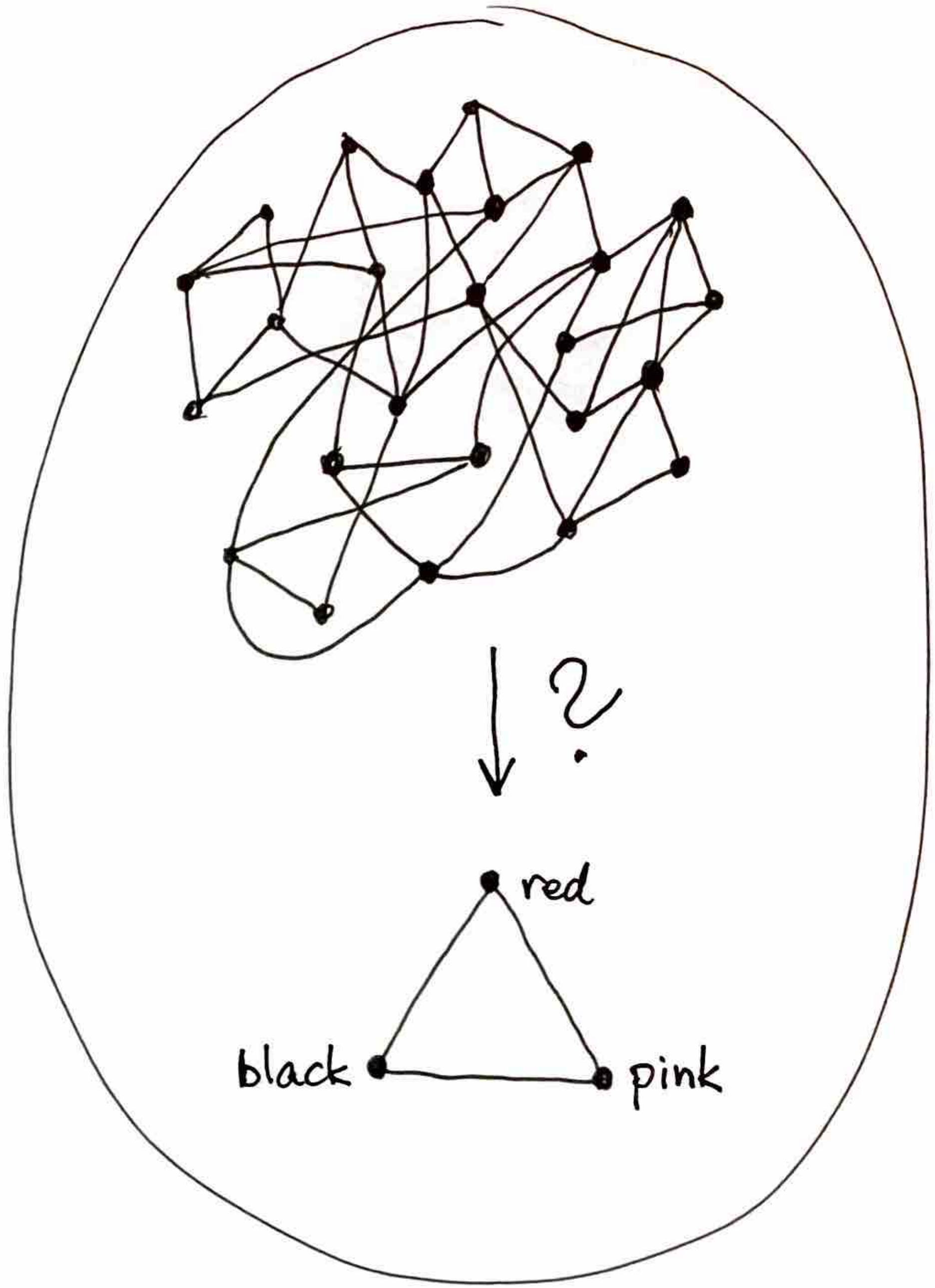
Problems $CSP(\mathbb{A})$

3-coloring problem

INPUT: graph $\mathbb{X}$

OUTPUT: $\mathbb{X} \longrightarrow \triangle$

(if it exists)



Question: how fast can it be solved?

CSP

$\mathbb{A}$: fixed digraph (or other structure)

- the fixed-template CSP
- many variants

CSP($\mathbb{A}$)

INPUT: finite digraph $\mathbb{X}$

OUTPUT: Find $\mathbb{X} \to \mathbb{A}$

- each $\mathbb{A} \rightsquigarrow$ computational problem
- how broad is this class?
    - general $\mathbb{A}$: all computational problems
    - finite $\mathbb{A}$: 3-coloring (), 3-SAT, LIN-$\mathbb{Z}_2$

always in NP

Why am I here?

3 REASONS

- broadening horizons
- established connections model theory $\leftrightarrow$ ∞-domain CSP
- ! more recent connections set theory $\leftrightarrow$ finite domain CSP

THEOREM (ZF) [de Bruijn, Erdős '51, Läuchli '71, Kátay, Tóth, Vidnyánszky '24, Tardiff]

For finite $\mathbb{A}$

- CSP($\mathbb{A}$) solved by arc-consistency $\Rightarrow$ Comp($\mathbb{A}$) true
  $\Rightarrow$ in P
- not $\uparrow$ $\Rightarrow$ Comp($\mathbb{A}$) implies $\exists S \subseteq \mathbb{R}^3$ non-measurable
- CSP($\mathbb{A}$) in P $\Rightarrow$ Comp($\mathbb{A}$) doesn't imply ultrafilter axiom
- not $\uparrow$ (P$\neq$NP) $\Rightarrow$ Com($\mathbb{A}$) implies — " —

Comp($\mathbb{A}$)

$\forall \mathbb{X}$

$\forall \mathbb{X}' \subseteq_{fin} \mathbb{X} \quad \mathbb{X}' \to \mathbb{A}$

$\Rightarrow \forall \mathbb{X} \to \mathbb{A}$

true assuming ultrafilter axiom

it's not a coincidence (some CSP results needed)

THEOREM [Todorčević, Vidnyánszky '21, Grebík, Vidnyánszky, Thornton '22]

For finite $\mathbb{A}$

- CSP($\mathbb{A}$) "very easy" $\Rightarrow$ Borel CSP($\mathbb{A}$) easy
- CSP($\mathbb{A}$) solved by local consistency $\overset{?}{\Rightarrow}$ Borel CSP($\mathbb{A}$) hard

$\mathrm{COMP}(\mathbb{K}_3) \Rightarrow$ ULTRAFILTER AXIOM

- $\mathcal{F}$ filter on X. Want $\mathcal{F} \subseteq \mathcal{U}$ ultrafilter

- graph $\mathbb{G}$ vertex set $\{0,1,2\}^X$

  $(a_x)_{x\in X} \text{ — } (b_x)_{x\in X}$ iff $\{x;\ a_x \neq b_x\} \in \mathcal{F}$

$\forall$ finite $\mathbb{G}' \subseteq \mathbb{G}$ is 3-colorable

$\Rightarrow \exists c: \{0,1,2\}^X \longrightarrow \{0,1,2\}$ 3-coloring of $\mathbb{G}$ (For $\mathcal{F} = \{X\}$ $=$ X-ary polymorphism of $\mathbb{K}_3$)

WLOG $c(00\ldots0)=0$, $c(11\ldots1)=1$, $c(22\ldots2)=2$

- $U \in \mathcal{U}$ iff $c(000\ldots\underbrace{1111}_{U}00\ldots0)=1$

Story about
logic & CSP

OUTLINE

~100 years ago [Post]
Boolean clones

~50 years ago [Schaefer]
Boolean CSPs

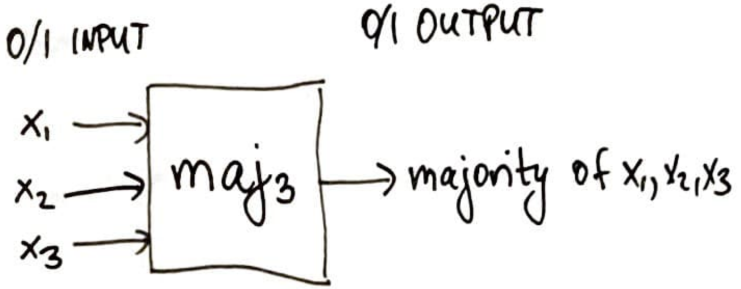
~25 years ago [Feder, Vardi
Jeavons]
tight connection

~20 years ago [Bulatov, Jeavons, Krokhin]
abstract clones

~5 years ago [Barto, Opršal, Pinsker
B, Bulín, Krokhin, Opršal]
minions

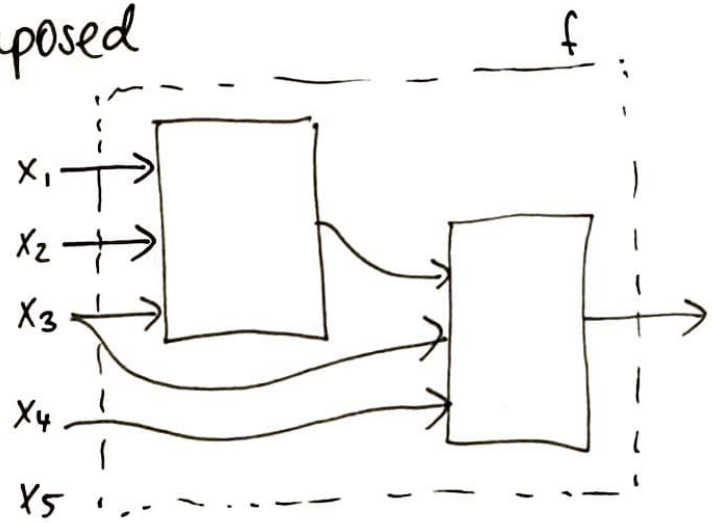
I. BOOLEAN CLONES : EXAMPLE

- consider



$$\{0,1\}^3 \longrightarrow \{0,1\}$$

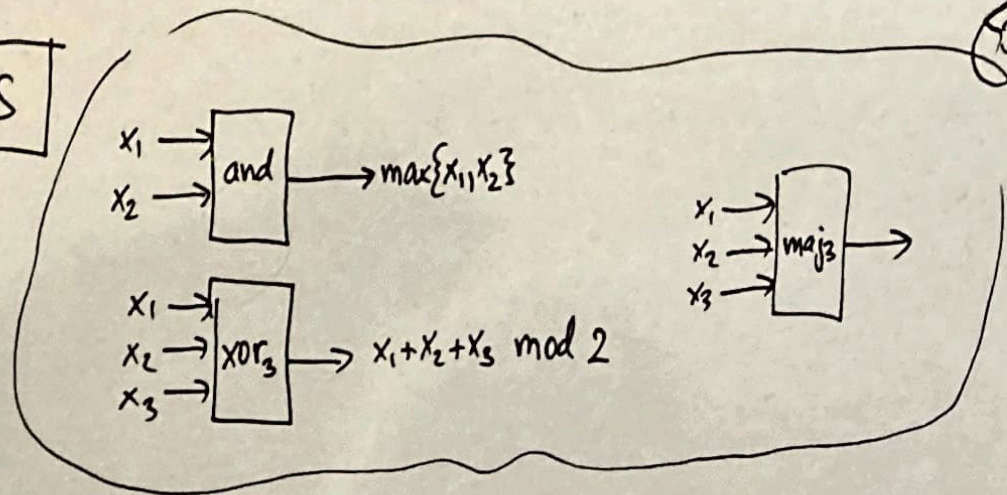
- can be composed



$$f(x_1, x_2, x_3, x_4, x_5) = maj_3(\ maj_3(x_1, x_2, x_3), x_3, x_4\ )$$

- – Can you compose maj_7 from maj_3? (maj_{37}, $maj_{1234567}$?)
  – what operations $\{0,1\}^n \to \{0,1\}$ can be composed from maj_3?

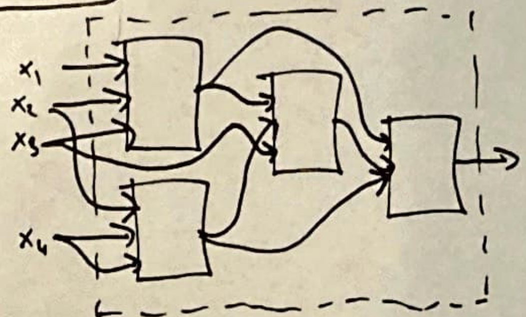
I. BOOLEAN CLONES : DEFINITIONS



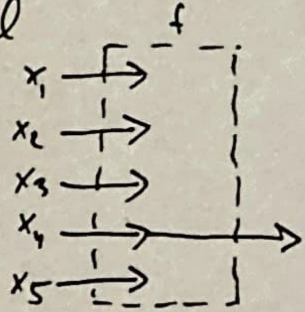
- **Boolean operation** $\{0,1\}^n \to \{0,1\}$

  (think: logical connective)

- **composition**



includes the trivial



$f(x_1, x_2, x_3, x_4, x_5) = x_4$

- **Boolean clone** set of Boolean operations closed under composition

- $\mathcal{F}$ set of Boolean operations

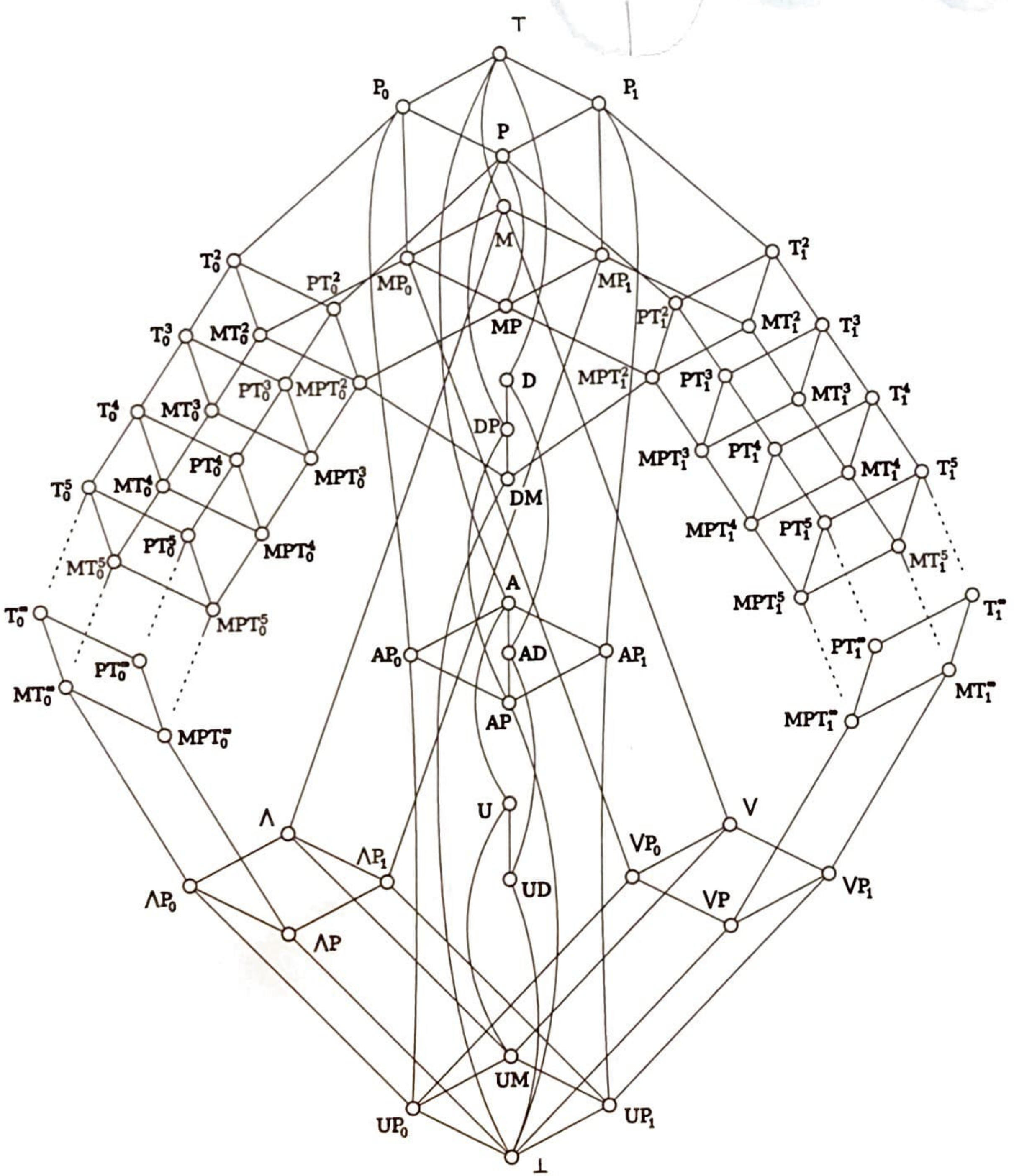
  $\mathrm{Clo}(\mathcal{F})$ = the smallest clone containing $\mathcal{F}$

  - $\mathrm{Clo}(\mathcal{F}) = \mathrm{Clo}(\mathcal{G})$ ... $\mathcal{F}, \mathcal{G}$ equally expressive
  - $\mathrm{Clo}(\mathcal{F}) \subset \mathrm{Clo}(\mathcal{G})$ ... $\mathcal{G}$ more expressive than $\mathcal{F}$

last slide

- $\mathrm{maj}_7 \in \mathrm{Clo}(\{\mathrm{maj}_3\})$?
- what is $\mathrm{Clo}(\{\mathrm{maj}_3\})$?

I. BOOLEAN CLONES – CLASSIFICATION



[Post '41 but...]

- poset
  (Boolean clones, $\subseteq$)
- = poset
  (sets of logical connectives, $\leq$ expressive power)
  / equal expressive power

I. GENERAL CLONES

- Clone on a set A

  set of operations on A ($A^n \to A$)
  closed under composition

- Dream: classify all clones

  More realistic: at least on $\{0,1,2\}$ ($2^{\aleph_0}$ many)

  More realistic: prove at least something

- Motivation

  - logic
  - algebra (algebras $\mathbb{A}, \mathbb{B}$ have the same clone of term operations $\Rightarrow$ they are "the same" for most purposes)

  - computational complexity...

I. BOOLEAN CSPs

Constraint Satisfaction Problems

- Boolean relation $R \subseteq \{0,1\}^n$

$R = \{0,1\}^2 \setminus \{(0,0)\}$ $R(x,y) \equiv x \vee y$

$S = \{0,1\}^3 \setminus \{(1,0,0)\}$ $S(x,y,z) \equiv \neg x \vee y \vee z$

- given Γ ... set of Boolean relations

$\Gamma = \{R, S\}$

CSP(Γ)

INPUT: pp-**sentence** over Γ

only $\exists, \wedge, R \in \Gamma$ allowed

OUTPUT: true?

$\exists x, y, z, u \;\; R(x,y) \wedge S(z,x,u) \wedge R(x,u)$

YES $x, y, z, u \mapsto 1$

- Examples: 2-SAT, HORN-SAT, 3-SAT, 3-COLORING, LINEAR EQ. OVER GF(2)
- THEOREM [Schaefer '78] $\forall \Gamma$ CSP(Γ) < in P / NP-complete
- Main tool: Γ pp-definable from Δ $\Rightarrow$ CSP(Γ) $\leq$ CSP(Δ)

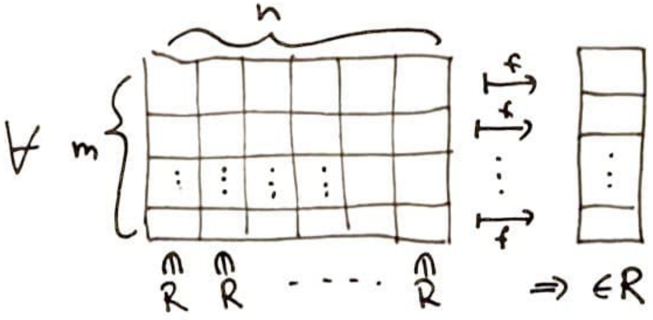
I. CLONES $\leftrightarrow$ CSPs : HISTORY

- [Schaefer '78] Post did something about Boolean operations, seems unrelated...
- [Feder, Vardi '98] Complexity of CSP(Γ) seems related to „closure properties" of relations in Γ
- [Jeavons '00] Complexity of CSP(Γ) IS related to —— " —— (by a result from 60s) $\rightsquigarrow$ clones
- [universal algebraists]

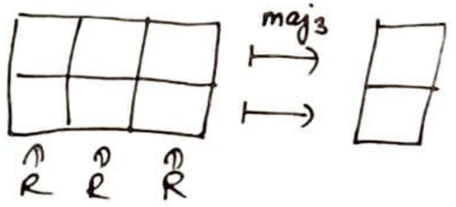
LET'S GO !!!

I. CLONES $\leftrightarrow$ CSPs : POLYMORPHISMS

- $f : \{0,1\}^n \to \{0,1\}$ is a polymorphism of $R \subseteq \{0,1\}^m$ if



maj_3 is a polymorphism of every $R \subseteq \{0,1\}^2$



- Γ ... set of Boolean relations

$\text{Pol}(\Gamma)$ ... polymorphisms of every $R \in \Gamma$. $\boxed{\text{Clone}}$

$\text{Pol}(\text{all binary}) = \text{Clo}(\text{maj}_3)$

- THEOREM [Geiger'68, Bodnarchuk, Kaluzhnin, Kotov, Romov'69]

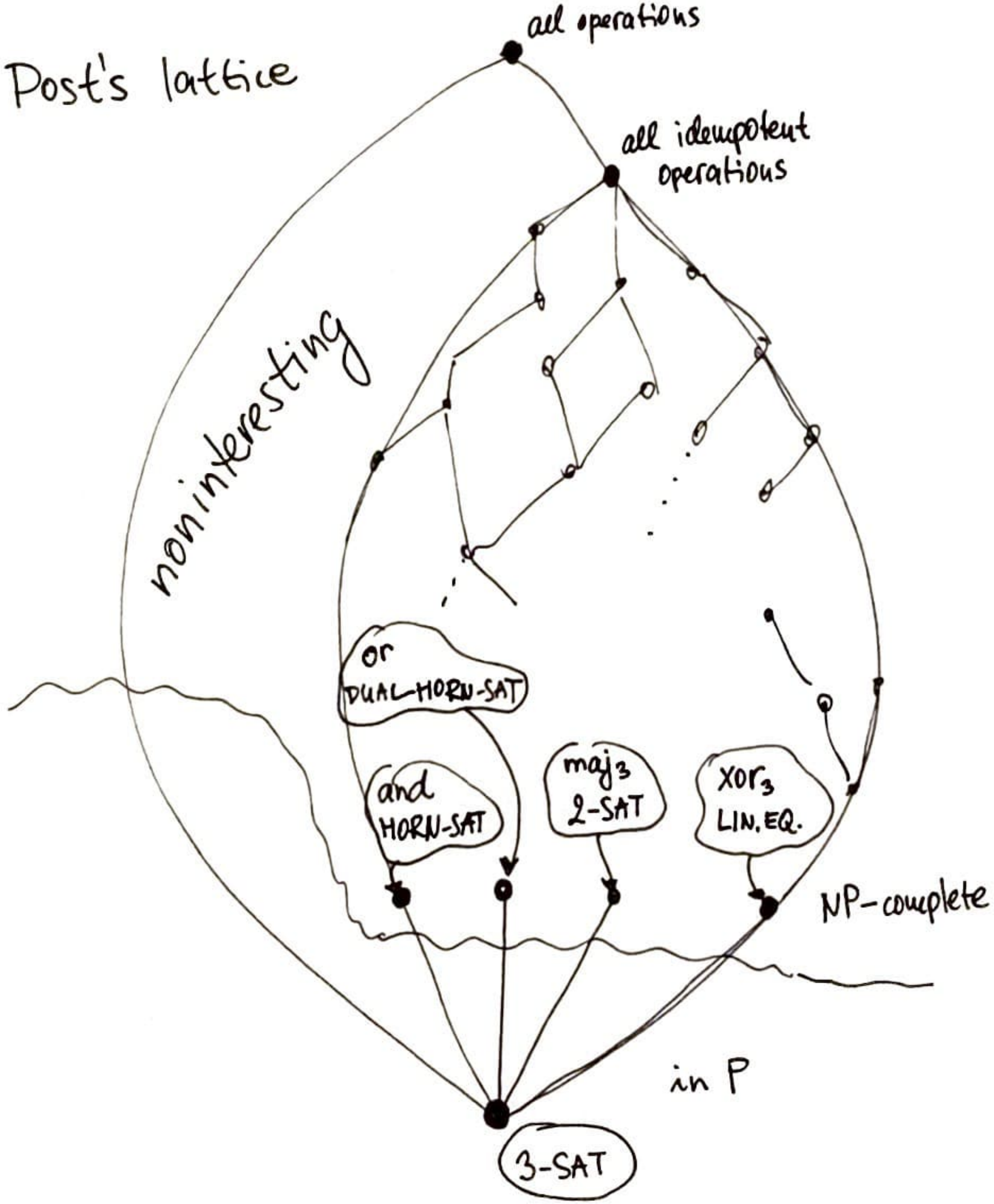
Γ pp-definable from $\Delta \iff \text{Pol}(\Gamma) \supseteq \text{Pol}(\Delta)$

(and then $\text{CSP}(\Gamma) \leq \text{CSP}(\Delta)$)
[Schaefer]

not only Boolean
... any finite A

I. CLONES ↔ CSPs: SCHAEFER REVISITED

Post's lattice



- 60s theorem implies

  Pol(Γ) higher
  ⇒ CSP(Γ) easier

- restrict to "interesting clones"
  - idempotent, i.e. $\forall a\ f(a,a,\dots,a)=a$

- 4 minimal clones by Post
  - Clo(and)
  - Clo(or)
  - Clo(maj_3)
  - Clo(xor_3)

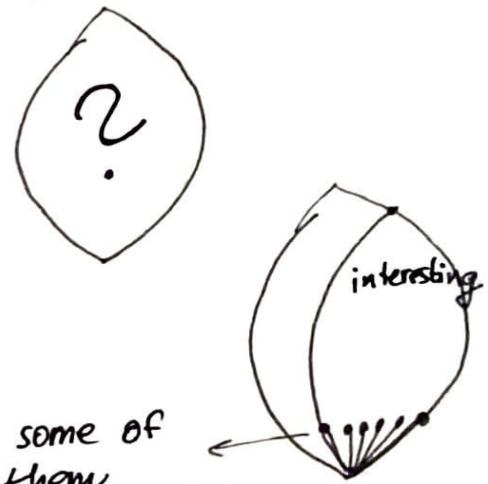
- they correspond to CSPs in P

  ⇒ Schaefer

I. NEW ORDERS

- 2 problems beyond Boolean
  1. classification hard even for clones on $\{0,1,2\}$
  2. for CSPs: borderline is not at the bottom

clones on $\{0,1,2\}$

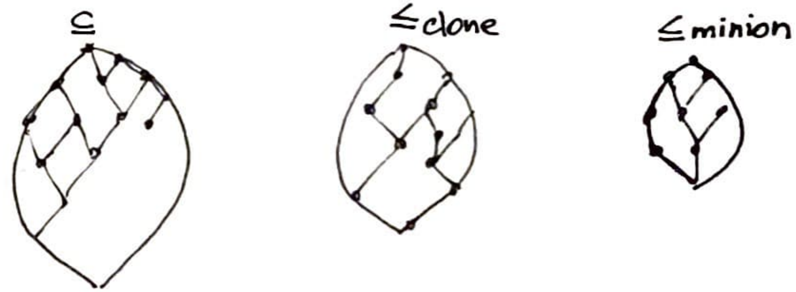


some of them NP-complete

- "better preorders" $\mathcal{C} \subseteq \mathcal{D} \Rightarrow \mathcal{C} \leq_{\text{clone}} \mathcal{D} \Rightarrow \mathcal{C} \leq_{\text{minion}} \mathcal{D}$

  ⇝ "simpler" posets

  still good for CSPs



⇝ "simpler" objects than clones

(here "simpler" = carries less information)

I. NEW ORDERS: DETAILS

(my life ① minions ② algebras, clones, CSPs ③ minions)

$CSP(\Gamma) \leq CSP(\Delta)$ if

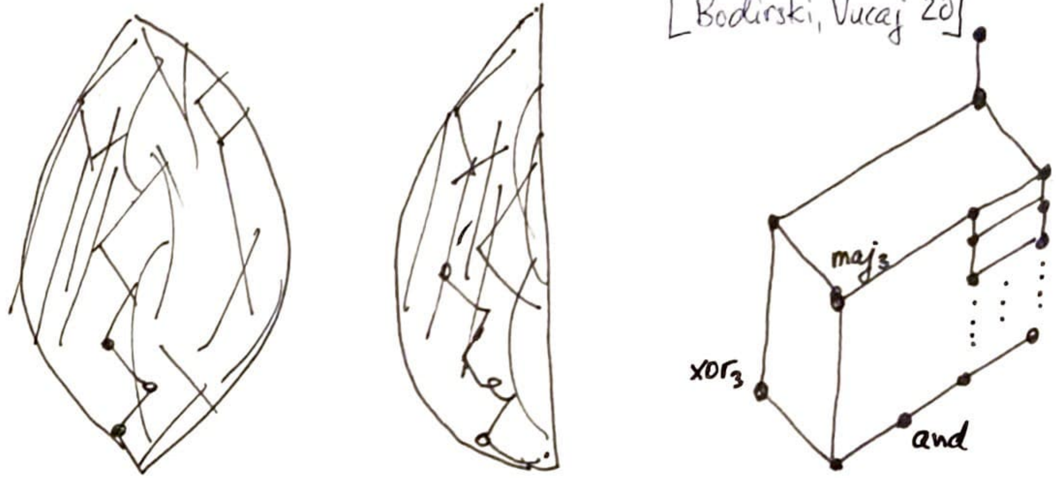
relationally	Γ pp-definable from Δ	Γ pp-interpretable in Δ	Γ pp-constructible from Δ
algebraically	$Pol(\Delta) \subseteq Pol(\Gamma)$	$\exists\, Pol(\Delta) \to Pol(\Gamma)$ **clone homomorphism** preserves arities & composition	$\exists\, Pol(\Delta) \to Pol(\Gamma)$ **minion homomorphism** preserves arities & permutation, identification of variables
preorder on clones	$\mathcal{C} \subseteq \mathcal{D}$	$\mathcal{C} \leq_{clone} \mathcal{D}$ if $\exists$ clone homo $\mathcal{C} \to \mathcal{D}$	$\mathcal{C} \leq_{minion} \mathcal{D}$ if $\exists$ minion homo $\mathcal{C} \to \mathcal{D}$
algebraic object	clone [Geiger'68; Bodnarchuk, Kaluzhnin Kotov, Romov'69 Jeavons'00]	abstract clone [Birkhoff'35 Bulatov, Jeavons, Krokhin'05 Bodirsky'08]	**minion** functor finite ordinals $\to$ sets [B, Opršal, Pinsker'18 B, Bulín, Krokhin, Opršal'21]

* see my life

I. NEW ORDERS: GAINS

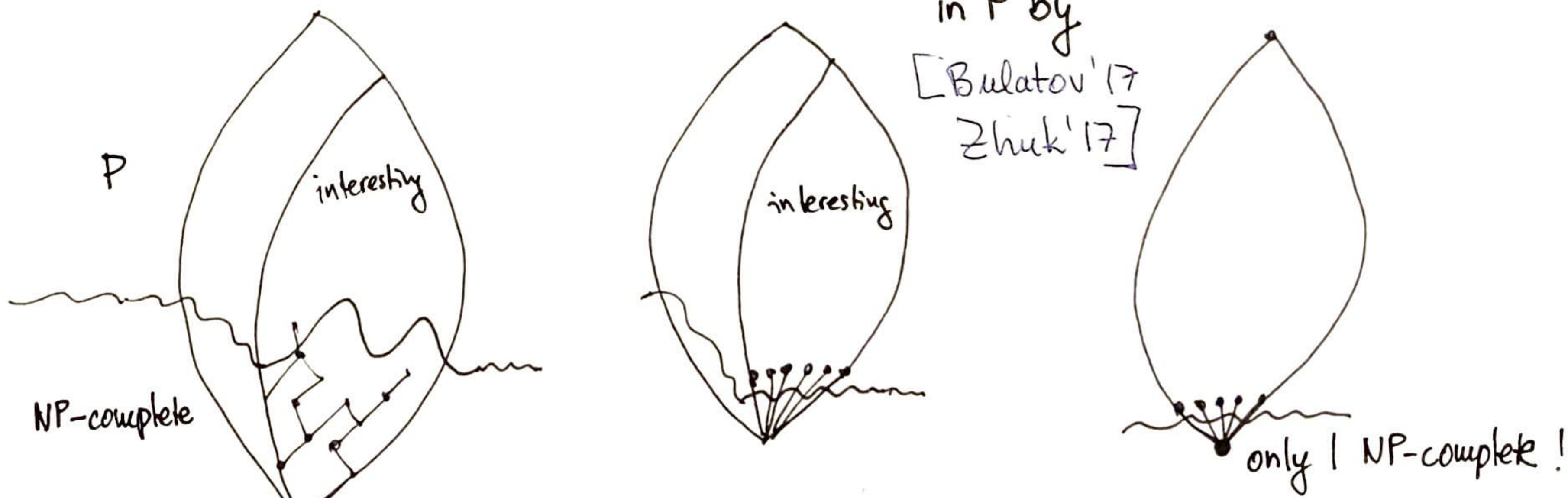
- Boolean clones <u>get</u> simpler

[Bodirski, Vucaj '20]



≤   ≤clone   ≤minion

- new hope for {0,1,2} (©Zhuk) or even bigger, e.g. possibly countable!
- beautiful borderlines for CSPs



III. WRAP UP

- clones $\to$ abstract clones $\to$ (abstract) minions $\to$?
- minions important for Promise CSPs

  e.g. 6-color a 3-colorable graph
- math: very infinite, infinite, finite, very finite

  active & cool part of logic & algebra

- object G $\to$ Aut(G) group : symmetries of G

  G $\to$ Pol(G) clone : multivariate symmetries of G

  - useful in CSPs
  - will it become a standard concept in math?

Thank you for listening!