On the measurable Hall condition IV

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Equidecompositions

Γ a set of isometries of $\mathbb{R}^n$. $A, B \subseteq \mathbb{R}^n$ are Γ-equidecomposable if there are finite partitions $A = \cup_{n=1}^{k} A_n$, $B = \cup_{n=1}^{k} B_n$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $A_i = \gamma_i B_i$.

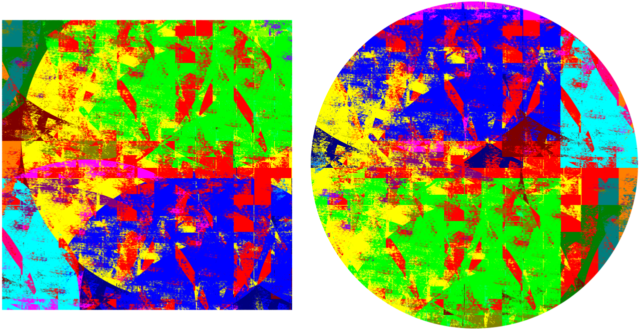
Banach, Tarski (1924): Any two bounded sets of nonempty interior in $\mathbb{R}^3$ are equidecomposable. In particular, the unit ball and the union of its two disjoint copies are equidecomposable

Circle squaring

Tarski (1925): Are the unit square and the disc of unit area equidecomposable by isometries?

Laczkovich (1990): Yes! By random translations.

Grabowski, Máthé, Pikhurko (2017): Measurable pieces

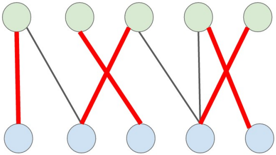


Marks, Unger (2017): Borel equidecomposition

Máthé, Noel, Pikhurko (2021): Boolean combinations of F_σ sets

From equidecompositions to perfect matchings

A perfect matching in a graph G is a set of edges such that every vertex is incident to exactly one of them.



Given a set of isometries Γ, A and B admit a Γ-equidecomposition iff the bipartite graph (=no odd cycle) with vertex set $V(G) = A \cup^* B$, and edge set $E(G) = \{(a,b) : \exists \gamma \in \Gamma \ \gamma a = b\}$ admits a perfect matching Γ.

Hyperfinite graphings

A graphing is a Borel graph over a standard probability measure space whose edge set is the countable union of the graph of measure-preserving bijections.

A graphing is (measurably) hyperfinite if the connectivity relation is hyperfinite (on a conull set), i.e., a countable increasing union of finite relations. Equivalently, for every $\varepsilon > 0$ after the removal of a set of edges of measure ε it has only finite components left.

Connes, Feldmann, Weiss (1981): Every pmp group action of an amenable group is measurably hyperfinite.

Fractional perfect matchings and ends

A fractional perfect matching in a graph G is a mapping $\tau : E(G) \to [0,1]$ such that $\sum_{y:(x,y)\in E(G)} \tau((x,y)) = 1$ for every $x \in V(G)$.

Every locally finite hyperfinite graphing with a PM admits a measurable fractional PM!

Adams (1990):Hyperfinite graphings have at most two ends a.e.

A hyperfinite graphing has zero ends a.e. iff the the components are finite, two ends iff it has linear growth a.e. and one end a.e. iff it has superlinear growth. (The growth at vertex x is $r \mapsto |B(x,r)|$.)

Measurable perfect matchings in hyperfinite graphings

Bowen, K, Sabok '21: A regular hyperfinite bipartite graphing admits a measurable perfect matching if it is one-ended or the degree is odd.

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Measurable perfect matchings in hyperfinite graphings

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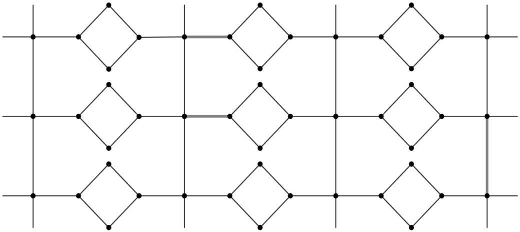
It is not enough that G is nowhere two-ended, see the next slide.

Why should we look at the MFPM?

Consider a graphing whose orbits are isomorphic to the following graph (can be obtained by the surgery of a pmp free $\mathbb{Z}^2$-action). The choice on certain edges can be forced: 4-cycles should contain two edges of every PM, and no edge connecting a 4-cycle and a vertical line can can be in a PM.



Applications

- Settling the question of Lyons-Nazarov for amenable groups: bipartite Cayley graph has a factor of iid PM iff the group is not the semidirect product of $\mathbb{Z}$ and a finite group of odd size. (Two-ended amenable groups are such semidirect products.)

- Measurable circle squaring: equidecompositions by two independent sets of translations give a measurable equidecomposition.

- "Finally a real application of this abstract nonsense!"
  Timár (2021): Factor PM of optimal tail between Poisson processes (improving Benjamini-Lyons-Peres-Schramm)

- Measurable balanced orientation in a one-ended graphing

Expansion in (bipartite) graphings

Assume $\lambda(N(S) \setminus S) > \varepsilon\lambda(S)$ if $\lambda(S) \leq \frac{1}{2}$.
Includes pmp ergodic actions of Kazhdan Property (T) groups.

Banach-Ruziewicz problem: For $n > 1$ is the only $SO(n)$-invariant finitely additive probability measure on S^n the Lebesgue measure?

Margulis (1980): $n \geq 5$

Sullivan (1981): $n \geq 4$

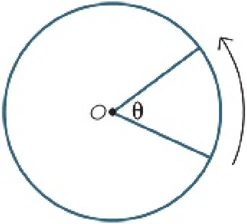
Drinfeld (1984): $n \geq 2$

Lyons, Nazarov (2011): Every bipartite Cayley graph of a non-amenable group admits a factor of iid perfect matching.

Grabowski, Máthé, Pikhurko (2017): $n \geq 3, A, B \subseteq \mathbb{R}^n$ bounded measurable of nonempty interior, $\lambda(A) = \lambda(B)$. Then A and B are measurably equidecomposable.

Graphings without measurable PM

Laczkovich (1988): 2-regular acyclic graphing without MPM.



Conley, Kechris (2013): Modify it to d-regular for even d.

An (essentially) acyclic graphing is called a treeing.

Marks (2013): d-regular treeing without Borel PM for $d > 2$.

Kechris, Marks (2018): Does every 3-regular treeing admit MPM?

Couplings, permutons, doubly stochastic measures

Given a measure μ on $[0,1]^2$ let μ^1, μ^2 be its marginals defined by $\mu^1(S) = \mu(S \times [0,1]), \mu^2(S) = \mu([0,1] \times S)$

Conjecture (Gurel-Gurevich, Peled '13): For every probability measure μ on $[0,1]^2$ if $\mu^1 = \mu^2 = \lambda$ and its sections are atomless then its support contains a.e. the graph of a measure-preserving bijection of $[0,1]$, i.e., the support of a determistic coupling.

Proved in special cases related to Poisson thickening.

Losert gave an example of an extreme point in the set of permutons with full support.

Markov spaces

Double measure space (Lovász): $(J, \mathcal{A}, \lambda, \eta)$, the node measure λ is a probability measure on the set of vertices J, which is the standard Borel space $(J, \mathcal{A})$, and the edge measure η is a symmetric measure on $(J \times J, \mathcal{A} \otimes \mathcal{A})$. Markov space if $\eta^1 = \eta^2 = \lambda$.
Common generalization of graphings and graphons.

A circulation measure α is a signed measure on $\mathcal{A}^2$ s.t. $\alpha(S \times J) = \alpha(J \times S)$, i.e., $\alpha = \alpha^T$.

Hoffmann's circulation theorem, MFMC... hold for Markov spaces.

$F : J \times J \to \mathbb{R}$ potential if $\exists\ p : J \to \mathbb{R}$ s.t. $F(x, y) = p(x) - p(y)$.

Lemma: α is a circulation iff $\int_{J \times J} F d\alpha = 0$ for every bounded measurable potential F.

Main results

K: For every $d > 2$ there exists a measurably bipartite, d-regular treeing without antisymmetric circulation. In particular, it has no MPM.

K: For every d there is a free pmp action of $\mathbb{Z}_2^{*d}$ without circulation. No free $\mathbb{Z}$-action on a subset of positive measure and is not the Schreier graphing of a free pmp action of $\mathbb{F}_{d/2}$.

K: There exists an atomless coupling μ on $[0,1]^2$ s.t. $supp(\mu)$ does not contain the support of any other coupling.
In particular, it does not contain the graph of a deterministic coupling refuting the conjecture of Gurel-Gurevich ad Peled.

Inverse limits of sequences of finite graphs

The inverse limit of $G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \ldots$ is the graph G with

vertex set $V(G) = J = \left\{(x_n)_{n=1}^{\infty} : \forall n \; x_n \in V(G_n), f_n(x_{n+1}) = x_n\right\}$,

and edge set $E(G) = \{(x, y) : x, y \in J, \forall n \; (x_n, y_n) \in E(G_n)\}$.

We endow J with the topology inherited from the product topology of the discrete topologies on $V(G_n)$.

From proper sequences to Markov spaces

A sequence of graphs is proper if for every n

1. $|f_n^{-1}(u)| = \frac{|V(G_{n+1}|}{|V(G_n)|}$ for every n and $u \in V(G_n)$,
2. $|f_n^{-1}(u,v) \cap E(G_{n+1})| = \frac{|E(G_{n+1})|}{|E(G_n)|}$,
3. every graph G_n is regular with degree at least two, and
4. every G_n is bipartite.

Set $\lambda\big(\{x : x_n \in S\}\big) := \frac{|S|}{|V(G_n)|}$ for every n and $S \subseteq V(G_n)$, this uniquely extends to the σ-algebra $\mathcal{A}$.
Define η by $\eta\Big(\{(x,y) : (x_n, y_n) \in Q\}\Big) := \frac{|Q \cap E(G_n)|}{|E(G_n)|}$
for every n and $Q \subseteq V(G_n)^2$, this uniquely extends to $\mathcal{A}^2$.

Basic properties of the inverse limit of a proper sequence

1. $V(G)$ is a Polish space, $E(G)$ is a closed subset of $V(G) \times V(G)$.
2. $(J, \mathcal{A}, \lambda)$ is a probability measure space.
3. $(J^2, \mathcal{A}^2, \eta)$ is a probability measure space and $\eta(J^2 \setminus E(G)) = 0$.
4. $\mathcal{D} = (J, \mathcal{A}, \lambda, \eta)$ is a double measure space.
5. $\eta^1 = \eta^2 = \lambda$, in other words, $\mathcal{D}$ is a Markov space.
6. If for some d every G_n is d-regular then $(J, \mathcal{A}, \lambda, d\eta)$ is a d-regular graphing.
7. G admits a clopen bipartition.

How to avoid circulations?

Key lemma: $\{G_n, f_n\}_{n=1}^{\infty}$ proper, $(J, \mathcal{A}, \lambda, \eta)$ its inverse limit. Assume that $\forall \varepsilon > 0\ \exists N\ \forall n > N$ and any orientation $\mathcal{O} \in Ori(G_n)\ \exists p_{\mathcal{O}} : V(G_{n+1}) \to \mathbb{R}$ such that

1. $|p_{\mathcal{O}}(u) - p_{\mathcal{O}}(v)| \leq 1$ for every $(u, v) \in E(G_{n+1})$,
2. $|\{u \in V(G_{n+1}) : \exists v\ (u, v) \in E(G_{n+1}),\ p_{\mathcal{O}}(u) - p_{\mathcal{O}}(v) \neq \mathcal{O}(f_n(u), f_n(v))\}| < \varepsilon |V(G_{n+1})|$.

Then there is no non-zero antisymmetric circulation measure α on $\mathcal{A}^2$ s.t. $|\alpha|^1 \ll \lambda, |\alpha|^2 \ll \lambda$ and $|\alpha|(J \times J \setminus E(G)) = 0$.

Hence the only coupling on $J \times J$ supported in $E(G)$ is η, and $E(G)$ does not contain a.e. a pmp perfect matching.

The proof of the key lemma

We prove by contradiction, consider $\alpha \neq 0$. Since $|\alpha|^1 \ll \lambda, |\alpha|^2 \ll \lambda$, Radon-Nikodym gives $\varepsilon > 0$ that for $S \subseteq J$ if $\lambda(S) \leq \varepsilon$ then $|\alpha|(S \times J \cup J \times S) < \frac{\|\alpha\|}{3}$.

Hahn decomposition gives $\mathcal{O}' : J^2 \to \{-1, 1\}$ such that

$$\int_{J^2} \mathcal{O}'(x, y)\ d\alpha(x, y) = \|\alpha\|.$$

Since $|\alpha|(J \times J \setminus E(G)) = 0$, if n is large enough then there is an orientation $\mathcal{O}$ of $E(G_n)$ s.t.

$$\int_{J^2} \mathcal{O}(x_n, y_n)\ d\alpha(x, y) > \frac{2\|\alpha\|}{3},$$

and there is $p_{\mathcal{O}} : V(G_{n+1}) \to \mathbb{R}$ for ε by assumption.
$S = \{x : \exists v \in V(G_{n+1})\ (x_{n+1}, v) \in E(G_{n+1}),$
$\mathrm{p}_{\mathcal{O}}(x_{n+1}) - p_{\mathcal{O}}(v) \neq \mathcal{O}(f_n(x_{n+1}), f_n(v))\}.$

Wrapping up

On the one hand,

$$\int_{J^2} p_{\mathcal{O}}(x_{n+1}) - p_{\mathcal{O}}(y_{n+1})\ d\alpha(x,y) = 0,$$

since $(x,y) \mapsto p_{\mathcal{O}}(x_{n+1}) - p_{\mathcal{O}}(y_{n+1})$ is a potential.

On the other hand, $\int_{J^2} p_{\mathcal{O}}(x_{n+1}) - p_{\mathcal{O}}(y_{n+1})\ d\alpha(x,y) \geq \int_{J^2} \mathcal{O}(x_n, y_n)\ d\alpha(x,y) - \int_{S\times J \cup J\times S} 2\ d|\alpha|(x,y)$.

This leads to a contradiction, since $\lambda(S) \leq \varepsilon$, and hence $\int_{J^2} \mathcal{O}(x_n, y_n)\ d\alpha(x,y) - \int_{S\times J \cup J\times S} 2\ d|\alpha|(x,y) > \frac{2\|\alpha\|}{3} - 2|\alpha|(S) = 0$, a contradiction.

The main building block of the constructions

G finite, d-regular graph and $N \in \mathbb{N}$. Define $F = F(G, N)$.

$V(F) = V(G) \times \Pi_{\mathcal{O} \in Ori(G)}[N]$

$f : V(F) \to V(G)$ and $p_{\mathcal{O}} : V(F) \to [N]$ for $\mathcal{O} \in Ori(G)$ denote the projections

$$E(F) = \{(x, y) : x, y \in V(F), (f(x), f(y)) \in E(G),$$
$$\forall \mathcal{O} \in Ori(G)\ p_{\mathcal{O}}(x) - p_{\mathcal{O}}(y) = \mathcal{O}(f(x), f(y))\}.$$

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$$E(F) = \{(x, y) : x, y \in V(F), (f(x), f(y)) \in E(G), \\ \forall \mathcal{O} \in Ori(G)\ p_{\mathcal{O}}(x) - p_{\mathcal{O}}(y) = \mathcal{O}(f(x), f(y))\}.$$

1. f is a graph homomorphism.
2. $\forall u\ deg(u) \leq d,\ deg(u) = d \iff \forall \mathcal{O}\ p_{\mathcal{O}}(u) \notin \{1, N\}$
3. $|\{u : deg(u) \lneq d\}| \leq \frac{2|Ori(G)|}{N}|V(F)|$
4. $|f^{-1}(u)| = N^{|Ori(G)|}$ for every $u \in V(G)$.
5. $|f^{-1}(u, v) \cap E(F)| = (N-1)^{|Ori(G)|}$ for every $(u, v) \in E(G)$.
6. $|\{s \in f^{-1}(u) : \nexists t \in f^{-1}(v), (s, t) \in E(F)\}| = N^{|Ori(G)|} - (N-1)^{|Ori(G)|}$ for every $(u, v) \in E(G)$.

The constructions

Graphings first! Set $G_1 = K_{d,d}$. In order to define G_{n+1} take d^2 copies of $F(G_n, 2^n)$ and a small number of additional vertices to make it regular. The mappings $f, p_{\mathcal{O}}$ can be extended if they are constant in the neighborhood of every additional vertex.

$V(G_{n+1}) = [d^2] \times V(F_n) \cup \{(x,j) : x \in V(F_n), j \in [d(d - deg_{F_n}(x))]\}$, and

$E(G_{n+1}) = \{((i,x),(i,y)) : (x,y) \in E(F_n), i \in [d^2]\} \cup \Big\{((i,x),(x,j)) : x \in V(F_n), \exists k \in [d - deg_{F_n}(x)], \ell \in [d], i(d - deg_{F_n}(x)) - k = jd - \ell\Big\}$.

For the coupling theorem construct G_{n+1} from G_n as above and take its direct product with an edge with loops. This doubles the degree in every step.

Questions

K: Which finitely generated groups admit a measurable perfect matching in their Schreier graphing of any pmp free ergodic action w.r.t. any finite symmetric set of generators?

Infinite Kazhdan, one-ended amenable (for bipartite Cayley graphs) do, certain two-ended groups do not. Free groups!?

K: Which finitely generated groups admit a measurable balanced orientation in the Schreier graphing of any pmp free ergodic action w.r.t. any finite symmetric set of generators?

Infinite Kazhdan, one-ended amenable do, certain two-ended groups do not. $(\mathbb{Z}/2\mathbb{Z})^{*d}$ does not with the standard set of generators, but with other generators it does! ($d \geq 2$)

K: Other LCL properties?

Thank you!