Categorical Partite Construction

Talk for the Winter School in Abstract Analysis

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31.01.2025

About this talk:

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- Introduction to the partite construction

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- Secret Mission: The nLab Agenda

Top Secret

Section 1

What are Small Ramsey Degrees?

Definition (Ramsey property).

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$$\left.\begin{array}{l} \forall A, B \in K, r \in \omega : \\ \exists C \in K : \\ \forall c \in r^{K(A,C)} : \\ \left.\begin{array}{l} \exists b \in K(B,C), k \in r : \\ \forall a \in K(A,B) : \\ c(b \circ a) = k. \end{array}\right\} \begin{array}{l} \exists b \in K(B,C) : \\ |c\,(b \circ -)\,[K(A,B)]| = 1 \end{array} \end{array}\right\} C \rightarrow (B)^A_{r,1}$$

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(2) *$A \in K$ has small Ramsey degree d if d is minimal such that*

$$\forall B \in K, r \in \omega : \exists C \in K : C \to (B)^A_{r,d}.$$

We say K has finite small Ramey degrees if all $A \in K$ have finite small Ramey degree.

Section 2

Categorical Languages

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- *for each co-function symbol $F \in L_F$ a co-function*

$$F^X \in \mathbf{Set}\left(K\left(X, \mathrm{ar}_{L_F}(F)(0)\right), K\left(X, \mathrm{ar}_{L_F}(F)(1)\right)\right).$$

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(5) *Analogously we define L-structures, where we have functions.*

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- *a relation for each* S^n *with arity* S^n *deciding and mapping to the homotopy type of contractability.*

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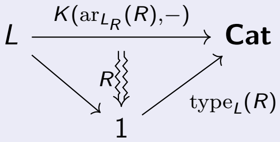

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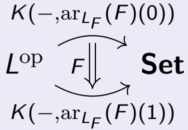

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- *which compose for $X, Y, Z \in K^R_\infty, \varphi \in K^R_\infty(X, Y)$, $\psi \in K^R_\mu(Y, Z)$:*

$$
\begin{aligned}
\psi \circ \varphi : K(X, Z) &\to \mathrm{type}_L(R) \\
h &\mapsto \bigvee_{\substack{f \in K(X,Y), \\ g \in K(Y,Z), \\ g \circ f = h}} \big(\psi(g) \wedge \varphi(f)\big).
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It comes with an inclusion $K \subseteq K^R_\mu$, defined by mapping morphisms f to the functor mapping only f to $\top$ and all parallel morphisms to $\bot$.

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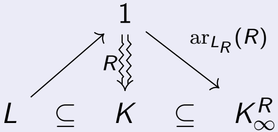

Lemma (Classification of languages II).

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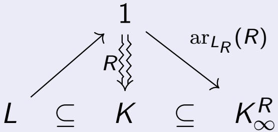
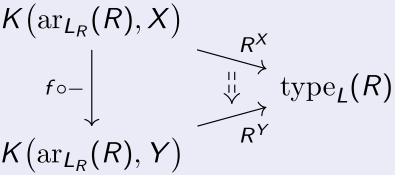


Proof.

$$\forall X, Y \in K, f \in K(X, Y):$$



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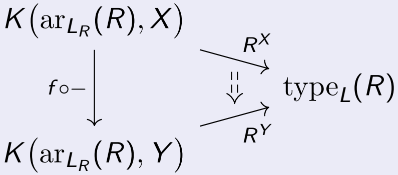

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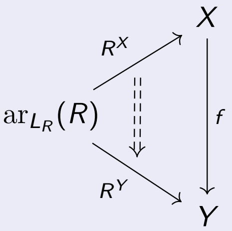

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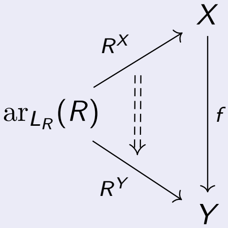

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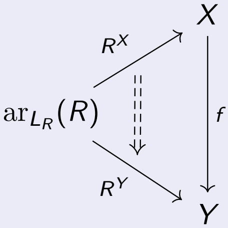

$\Updownarrow$ $\mathrm{type}_L(R)$ on-objects-surjectively embedds into a lattice and is hence a partial order, where everything commutes.

□

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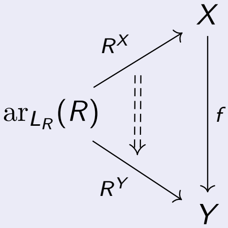
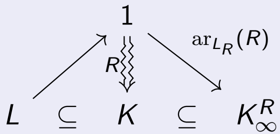


$\mathrm{type}_L(R)$ on-objects-surjectively embedds into a lattice and is hence a partial order, where everything commutes.



□

Section 3

Language colimits

Theorem (L-cross-structure colimits).

Let K be a category and L a K-cross-language.

Theorem (L-cross-structure colimits).

Let K be a category and L a K-cross-language.
Cocones in K of diagrams in L are (K-isomorphic to) cocones in L and being a colimit gets preserved.

Proof.

Let Z be a cocone of $G : J \to L \subseteq K$ in K.

$\square$

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- **"Relations"** For every relation symbol $R \in L_R$, the lax transformation, which is the associated lax cone in K_∞^R, can be extended to the colimit by vertical composition with the natural transformation that is the cone Z.

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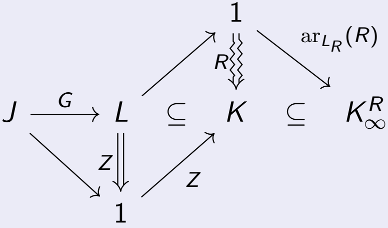

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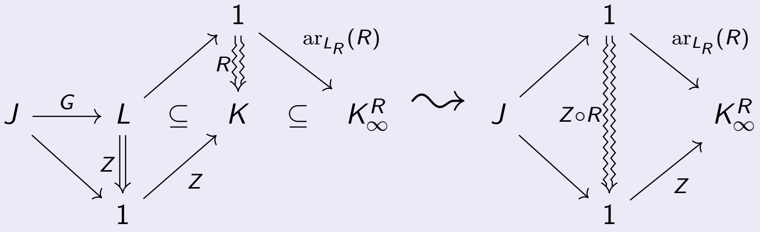

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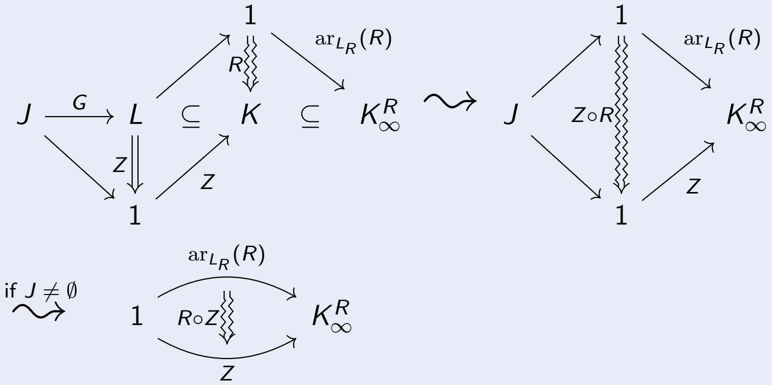

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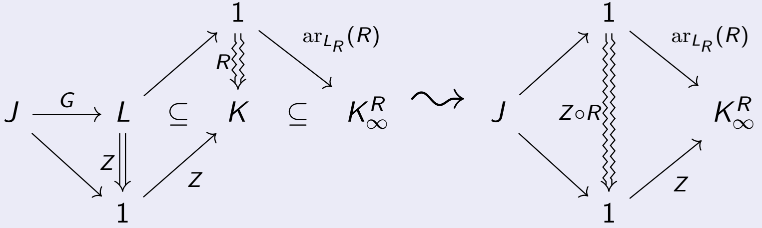

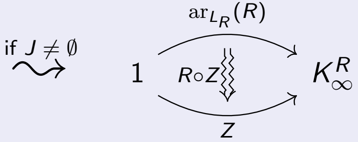

$\rightsquigarrow (R \circ Z)_0 \in K_\infty^R(\mathrm{ar}_{L_R}(R), Z)$

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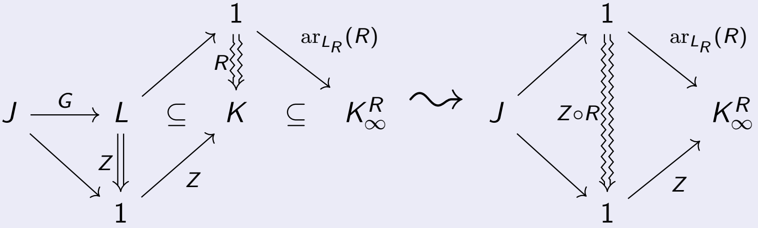
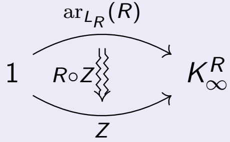


if $J \neq \emptyset$ $\rightsquigarrow$



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extends the lax cocone R.

Proof.

Let Z be a cocone of $G : J \to L \subseteq K$ in K.

"Cocone in L" First we define a cocone in L.

- **"Co-functions"** For any co-function symbol $F \in L_F$, the associated natural transformation can be extended to the cocone Z by vertically composing it with the opposite natural transformation corresponding to Z.

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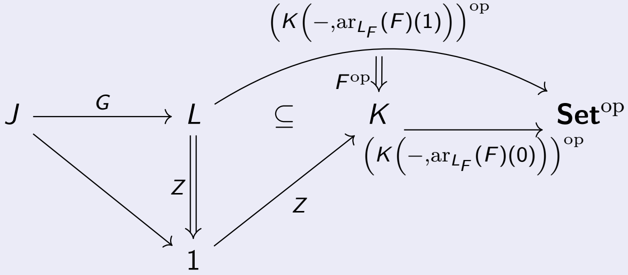

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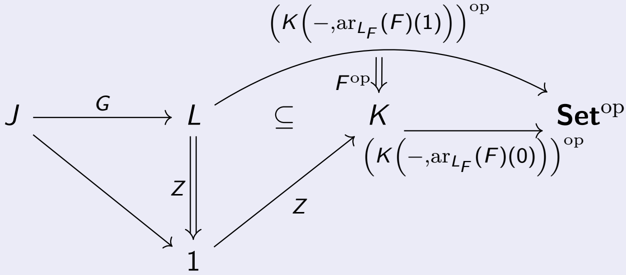

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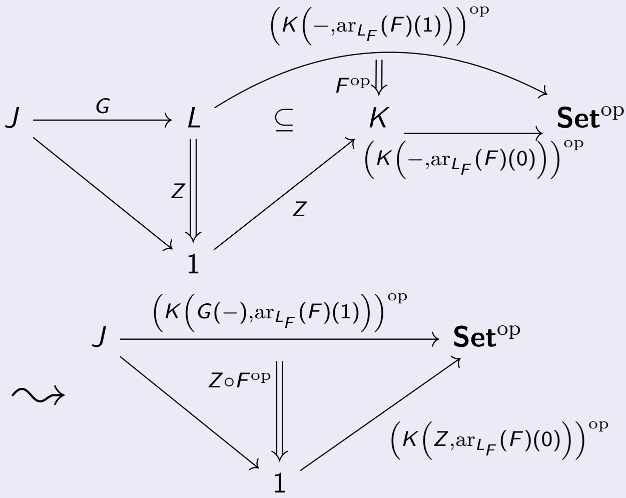

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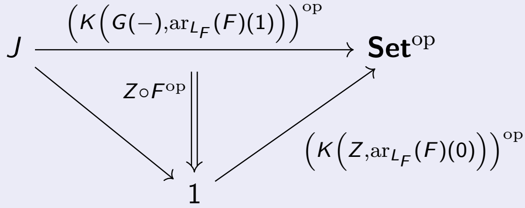

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$$\rightsquigarrow \quad \begin{array}{ccc} J & \xrightarrow{\left(K\left(G(-), \mathrm{ar}_{L_F}(F)(1)\right)\right)^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ & \searrow \quad \Downarrow Z \circ F^{\mathrm{op}} \quad \nearrow_{\left(K\left(Z, \mathrm{ar}_{L_F}(F)(0)\right)\right)^{\mathrm{op}}} & \\ & 1 & \end{array}$$

So the result written out is $\forall X, Y \in J, f \in J(X, Y)$:

$$\begin{array}{ccc} K\left(G(X), \mathrm{ar}_{L_F}(F)(1)\right) & & \\ \uparrow {\scriptstyle -\circ G(f)} & \circlearrowleft & \nwarrow {\scriptstyle \left(Z \circ F^{\mathrm{op}}\right)_X} \\ & & K\left(Z, \mathrm{ar}_{L_F}(F)(0)\right) \\ & & \swarrow {\scriptstyle \left(Z \circ F^{\mathrm{op}}\right)_Y} \\ K\left(G(Y), \mathrm{ar}_{L_F}(F)(1)\right) & & \end{array}$$

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Hence for all $X \in J$ the morphisms

$$\left(Z \circ F^{\mathrm{op}}\right)_X \in \mathbf{Set}\left(K\big(Z, \mathrm{ar}_{L_F}(F)(0)\big), K\big(G(X), \mathrm{ar}_{L_F}(F)(1)\big)\right)$$

extend the natural transformation F to the cocone Z.

Proof.

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"Preserving being a colimit" Now we show that the defined cocone preserves being initial in L.

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□

Section 4

Junges Theorem

Definition (Blocks).

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(2) *Let $f \in K(U, V)$ be a K-morphism, $\pi \in K(X, U)$ a split-monic domain object, $\rho \in K(Y, V)$ be a codomain object and $N \in \omega$.*

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(3) *The line diagram of J is the functor $G : J \to \mathrm{Bl}_f$ defined by $G(e \in l) :\in \mathrm{im}(e \setminus l) = \{\sigma \in \mathrm{Bl}_{i_0}(\pi, \rho) \mid l(\sigma) = e\}$.*

Theorem (Sebastian Junge; [Jun22, Thm. 1]).

Let K be a category that has colimits (over diagrams where the objects are lines in Bl_f), L a K-cross-language with only strong relations and $f \in K(U, V)$ be split-monic.

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Use Theorem for L-structure colimits

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Proof.

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Section 5

Ramsey Transfer

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Section 6

Hales-Jewett

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Let P be a finite set.
Then $\forall a, b, r \in \omega : \exists c \in \omega : c \to (b)^a_r$ in $\mathrm{HJ}(P)$.

Section 7

Partite Construction

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Theorem (Partite construction).

Let $G : D \to K$ be a functor between locally finite categories with split-monic image. Suppose K has finite colimits and D has a bound $b \in \omega + 1$ on its small Ramsey degrees. Let L be a K-cross-language.

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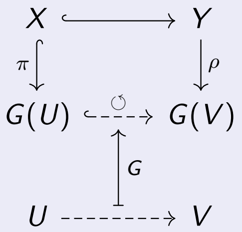

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"G-block ρ_0"

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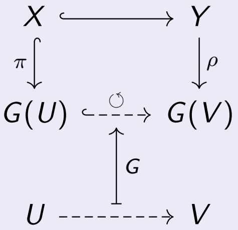

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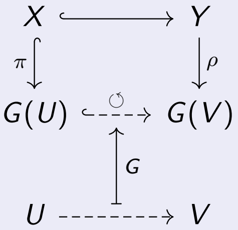

"G-block ρ_0"

By the Theorem for L-structure colimits the coproduct $Y_0 := \coprod_{D(V,W)} Y$ is a L-cross-structure. Note that for each $f \in D(V, W)$, incl_f is split-monic. $G(W)$ with K-morphisms $G(f) \circ \rho$ for $f \in D(V, W)$ forms a cocone, and thus there exists a unique commuting $\rho_0 \in K(Y_0, G(W))$.

Proof.

Let $\pi \in K(X, G(U))$, $\rho \in K(Y, G(V))$ be G-blocks where π is split-monic.
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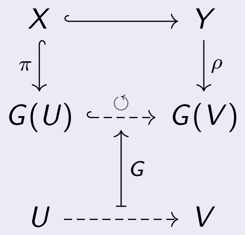
$$\forall r \in \omega : \exists W \in D, d \in b : W \to (V)^U_{r,d}.$$



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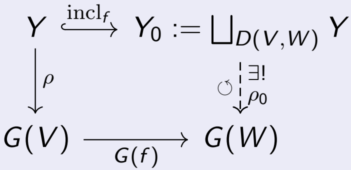
By the Theorem for L-structure colimits the coproduct $Y_0 := \bigsqcup_{D(V,W)} Y$ is a L-cross-structure. Note that for each $f \in D(V, W)$, incl_f is split-monic. $G(W)$ with K-morphisms $G(f) \circ \rho$ for $f \in D(V, W)$ forms a co-cone, and thus there exists a unique commuting $\rho_0 \in K(Y_0, G(W))$.

$\exists! \rho_0 \in K(Y_0, G(W)) : \forall f \in D(V, W) : \rho_0 \circ \mathrm{incl}_f = G(f) \circ \rho$.



□

Proof.

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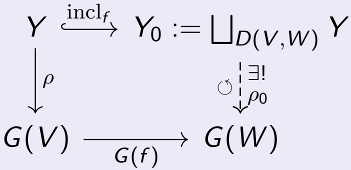

"G-block ρ_n" Enumerate $D(U,W) =: \{f_k \mid k \in n\}$.

Proof.

"*G*-**block** ρ_0"

By the Theorem for L-structure colimits the coproduct $Y_0 := \coprod_{D(V,W)} Y$ is a L-cross-structure. Note that for each $f \in D(V,W)$, incl_f is split-monic. $G(W)$ with K-morphisms $G(f) \circ \rho$ for $f \in D(V,W)$ forms a co-cone, and thus there exists a unique commuting $\rho_0 \in K(Y_0, G(W))$.

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$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{incl}_f} & Y_0 := \coprod_{D(V,W)} Y \\ \downarrow \rho & \circlearrowleft & \downarrow \exists! \rho_0 \\ G(V) & \xrightarrow[G(f)]{} & G(W) \end{array}$$

"*G*-**block** ρ_n" Enumerate $D(U,W) =: \{f_k \mid k \in n\}$. For $k \in n$ iteratively define from ρ_0, using the Partite Lemma for $\mathrm{Bl}^m_{G(f_k)}$,

$$\rho_{k+1} \to (\rho_k)^{\pi}_{r,1}.$$

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Define a coloring of $D(U, W)$ where the color of $f_k \in D(U, W)$ is the one corresponding to g_k for $\mathrm{Bl}^m_{G(f_k)}$.

Proof.

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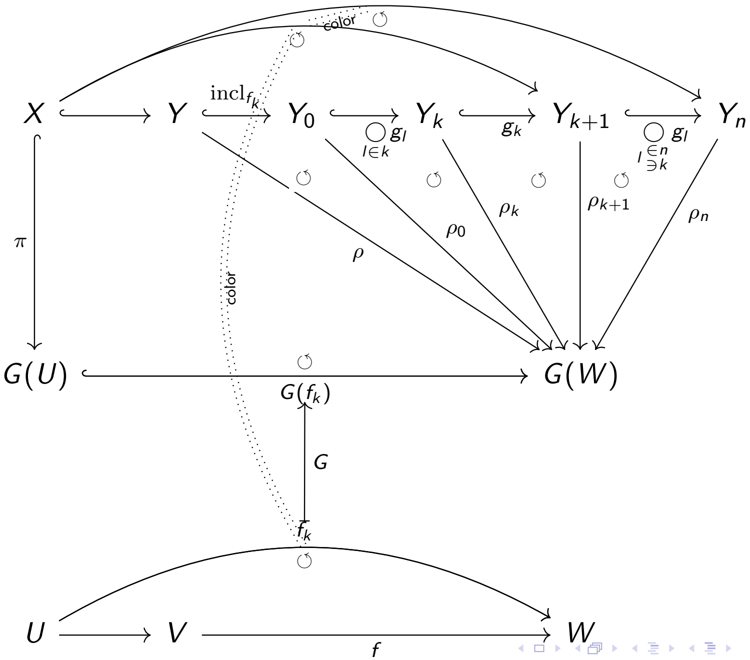
Define a coloring of $D(U,W)$ where the color of $f_k \in D(U,W)$ is the one corresponding to g_k for $\mathrm{Bl}^m_{G(f_k)}$.
Then, since $W \to (V)^U_{r,d}$, there is a d-chromatic $f \in D(V,W)$.
Thus, since $\mathrm{incl}_f \in \mathrm{Bl}^m_{G(f)}(G(V), G(W))$, $\bigcirc_{k \in n} g_k \circ \mathrm{incl}_f$ witnesses that $\rho_n \to (\rho)^{\pi}_{r,d}$.

□


X
Y
incl_{f_k}
Y_0
Y_k
Y_{k+1}
Y_n
$\bigcirc_{l \in k} g_l$
g_k
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color
color
π
ρ
ρ_0
ρ_k
ρ_{k+1}
ρ_n
$G(U)$
$G(W)$
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G
f_k
U
V
W
f

Thank you for your attention!

Sebastian Junge.
Categorical view of the partite lemma in structural ramsey theory.
Applied Categorical Structures, 31:1–13, 2022.