c_0-products of function spaces

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Theorem (Gul'ko, Khmyleva)

The spaces $c_0 \times s$ and c_0 are homeomorphic.

Definition

Let $\{(X_n, \| \ \|_n) : n \in \mathbb{N}\}$ let be a family of topological vector spaces equipped with a norm. The c_0-product of the family $\{(X_n, \| \ \|_n) : n \in \mathbb{N}\}$ is the space

$$\Big(\prod_{n\in\mathbb{N}} X_n\Big)_0 = \Big\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \|x_n\|_n \to 0\Big\}$$

endowed with the product topology.

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Here the topology of X_n does not need to be generated by the norm $\| \, \|_n$.

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For a pseudocompact space X we endow the space $C_p(X)$ with the supremum norm.

Theorem (Gul'ko, Khmyleva)

For a pseudocompact space X, the spaces $\left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ and $\prod_{n\in\mathbb{N}} C_p(X) \times \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ are homeomorphic.

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Proof.

Let $h : c_0 \to s \times c_0$ be a homeomorphism constructed by Gul'ko and Khmyleva. A sequence $(f_n)_{n\in\mathbb{N}} \in \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ is mapped to the pair of sequences $((g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}})$ such that, for every point $x \in X$,

$$h((f_n(x))_{n\in\mathbb{N}}) = ((g_n(x))_{n\in\mathbb{N}}, (h_n(x))_{n\in\mathbb{N}}).$$

Corollary (Gul'ko, Khmyleva)

Let X be a pseudocompact space. If $C_p(X) \sim \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$, then $C_p(X) \sim C_p(X)^\omega$.

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Problem

Is it true that $\left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0 \sim \left(\left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0\right)^\omega$ for any pseudocompact space X ?

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The answer is "yes" !

In the following lemma and theorem we consider the following norms

$$\|(f_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}}(\sup_{x\in X}(|f_n(x)|)), \text{ for } (f_n)_{n\in\mathbb{N}} \in \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0$$

and

$$\|(r,(f_n)_{n\in\mathbb{N}})\| = max\{|r|, \|(f_n)_{n\in\mathbb{N}}\|\}, \text{ for } (r,(f_n)_{n\in\mathbb{N}}) \in \mathbb{R}\times\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0.$$

Lemma

For a pseudocompact space X, there exists a linear homeomorphism

$$h : \mathbb{R} \times \Big(\prod_{n \in \mathbb{N}} C_p(X) \Big)_0 \to \Big(\prod_{n \in \mathbb{N}} C_p(X) \Big)_0$$

such that

$$\frac{1}{3} \|z\| \leq \|h(z)\| \leq 3 \|z\| .$$

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Proof.

h is given by the formula

$$h(r, (f_k)_{k\in\mathbb{N}}) = (f_1 - f_1(x_0) + r, (f_k - f_k(x_0) + f_{k-1}(x_0))_{k\geq 2})$$

and h^{-1} is given by the formula

$$h^{-1}((g_k)_{k\in\mathbb{N}}) = (g_1(x_0), (g_k - g_k(x_0) + g_{k+1}(x_0))_{k\geq 1}).$$

Theorem

For every pseudocompact space X, we have

$$\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \sim \Big(\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)^\omega$$

Theorem

For every pseudocompact space X, we have

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Proof (1/2).

Consider the space $Y = (\dot{\bigcup}_{n\in\mathbb{N}} X_n)\dot{\cup}\{a\}$, where $X_n \sim X$ for every $n \in \mathbb{N}$. The topology on $\dot{\bigcup}_{n\in\mathbb{N}} X_n$ is the topology of a disjoint union, and all open neighbourhoods of a consist of $\{a\}$ and cofinitely many copies of X. Notice that Y is pseudocompact as well.

Let $g : C_p(Y) \to \mathbb{R} \times \Big(\prod_{n\in\mathbb{N}} C_p(X_n)\Big)_0$ be given by the formula

$$g(f) = (f(a), (f\restriction_{X_n} - f(a))_{n\in\mathbb{N}}).$$

Then g is a linear homeomorphism, and $\frac{1}{2}\|f\| \leq \|g(f)\| \leq 2\|f\|$.

Proof (2/2).

Composing it with the homeomorphism from the previous Lemma we obtain a linear homeomorphism $p = h \circ g : C_p(Y) \to \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ is such that $\frac{1}{6}\|f\| \leq \|p(f)\| \leq 6\|f\|$ for every $f \in C_p(Y)$. The function

$$F : \Big(\prod_{n\in\mathbb{N}} C_p(Y)\Big)_0 \to \Big(\prod_{n\in\mathbb{N}} \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)_0$$

given by $F((f_n)_{n\in\mathbb{N}}) = (p(f_n))_{n\in\mathbb{N}}$ is a linear homeomorphism. Consequently,

$$\Big(\prod_{n\in\mathbb{N}} C_p(Y)\Big)_0 \sim \Big(\prod_{n\in\mathbb{N}} \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)_0 \sim \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \sim C_p(Y)$$

Using Corollary we obtain,

$$\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \sim C_p(Y) \sim C_p(Y)^\omega \sim \Big(\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)^\omega.$$

Thank you for your attention !