Some countable Rado-like graphs via Fraisse limits

Jarosław Swaczyna

Łódź University of Technology

Winterschool in Abstract Analysis 30.01.2025

(work in progress joint with Agnieszka Widz and Szymon Głąb)

Rado graph

Definition

Rado graph is unique countable graph $\mathcal{R}$ such that for any two finite disjoint $A, B \subset \mathcal{R}$ there is a vertex $v \in \mathcal{R}$ such that vEw holds for all $w \in A$ and no $w \in B$.

Equivalently

Rado graph has number of alternative descriptions

- Result of tossing a coin for every pair of elements of countable set;
- Fraisse limit of all finite graphs with all graph embeddings.
- And others.

Rado graph

Definition

Rado graph is unique countable graph $\mathcal{R}$ such that for any two finite disjoint $A, B \subset \mathcal{R}$ there is a vertex $v \in \mathcal{R}$ such that vEw holds for all $w \in A$ and no $w \in B$.

Equivalently

Rado graph has number of alternative descriptions

- Result of tossing a coin for every pair of elements of countable set;
- Fraisse limit of all finite graphs with all graph embeddings.
- And others.

Rado graph

Definition

Rado graph is unique countable graph $\mathcal{R}$ such that for any two finite disjoint $A, B \subset \mathcal{R}$ there is a vertex $v \in \mathcal{R}$ such that vEw holds for all $w \in A$ and no $w \in B$.

Equivalently

Rado graph has number of alternative descriptions

- Result of tossing a coin for every pair of elements of countable set;
- Fraisse limit of all finite graphs with all graph embeddings.
- And others.

Rado graph

Definition

Rado graph is unique countable graph $\mathcal{R}$ such that for any two finite disjoint $A, B \subset \mathcal{R}$ there is a vertex $v \in \mathcal{R}$ such that vEw holds for all $w \in A$ and no $w \in B$.

Equivalently

Rado graph has number of alternative descriptions

- Result of tossing a coin for every pair of elements of countable set;
- Fraisse limit of all finite graphs with all graph embeddings.
- And others.

NN^c-property

Rado graphs enjoys some quite rare properties:

- ultrahomogeneity;
- pigeonhole principle;

Definition

We say that graph G satisfies 1-NN^c property, if for every $v \in G$, both N_v, N_v^c are isomorphic with G.

Fact

Rado graph satisfies 1-NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

NN^c-property

Rado graphs enjoys some quite rare properties:

- ultrahomogeneity;
- pigeonhole principle;

Definition

We say that graph G satisfies 1-NN^c property, if for every $v \in G$, both N_v, N_v^c are isomorphic with G.

Fact

Rado graph satisfies 1-NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

NN^c-property

$$N_v = \{g \in G : gEv\},\ N_v^c = \{g \in G : g \neq v \wedge \neg gEv\}$$

Definition

We say that graph G satisfies 1-NN^c property, if for every $v \in G$, both N_v, N_v^c are isomorphic with G.

Fact

Rado graph satisfies 1-NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

NN^c-property

$$N_v = \{g \in G : gEv\}, N_v^c = \{g \in G : g \neq v \wedge \neg gEv\}$$

Definition

We say that graph G satisfies 1-NN^c property, if for every $v \in G$, both N_v, N_v^c are isomorphic with G.

Fact

Rado graph satisfies 1-NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

NN^c-property

$$N_v = \{g \in G : gEv\},\ N_v^c = \{g \in G : g \neq v \wedge \neg gEv\}$$

Definition

We say that graph G satisfies 1 -NN^c property, if for every $v \in G$, both N_v, N_v^c are isomorphic with G.

Fact

Rado graph satisfies 1 -NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

NN^c-property

$$N_v = \{g \in G : gEv\},\ N_v^c = \{g \in G : g \neq v \wedge \neg gEv\}$$

Definition

We say that graph G satisfies k -NN^c property, if for every disjoint $A, B \subset G$ with $\mathrm{card}(A \cup B) \leq k$ graph induced by $\bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$ is isomorphic with G.

Fact

Rado graph satisfies k -NN^c property.

Question, Bonato 2004

Let G be 1-NN^c graph. Is G necessarily isomorphic with the Rado graph?

Gordinowicz's answer

Theorem (Gordinowicz 2010)

There is 1-NN^c which is not 2-NN^c, therefore not isomorphic with the Rado graph. (construction)

Theorem (Andrzejczak, Gordinowicz 2013)

Characterization of finite isomorphism of above graph which extends to total automorphism.

Gordinowicz's answer

Theorem (Gordinowicz 2010)

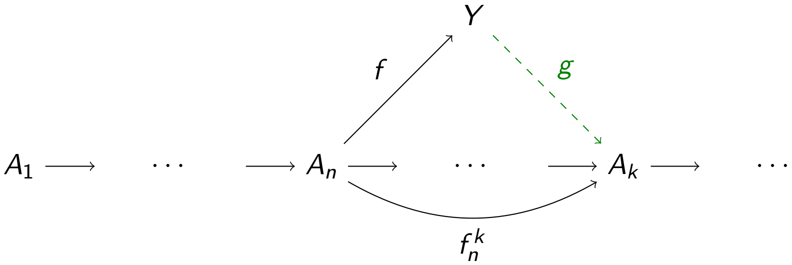
There is 1-NN^c which is not 2-NN^c, therefore not isomorphic with the Rado graph. (construction)

Theorem (Andrzejczak, Gordinowicz 2013)

Characterization of finite isomorphism of above graph which extends to total automorphism.

Fraisse limits (a'la Kubiś)

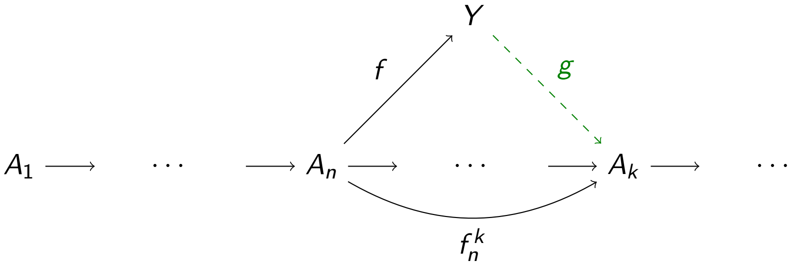
Given essentially countable directed category with amalgamation $\mathcal{K}$ there exists a unique Fraisse sequence:



In this presentation we may assume that $A_n \subset A_{n+1}$ and think of the graph $G = \bigcup_n A_n$.
Kubiś proved some results about uniqueness and homogeneity of such sequences.

Fraisse limits (a'la Kubiś)

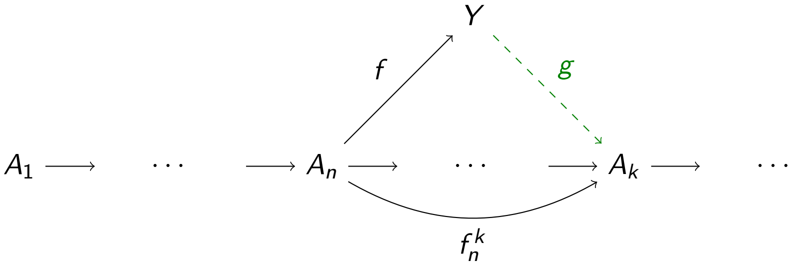
Given essentially countable directed category with amalgamation $\mathcal{K}$ there exists a unique Fraisse sequence:



In this presentation we may assume that $A_n \subset A_{n+1}$ and think of the graph $G = \bigcup_n A_n$.
Kubiś proved some results about uniqueness and homogeneity of such sequences.

Fraisse limits (a'la Kubiś)

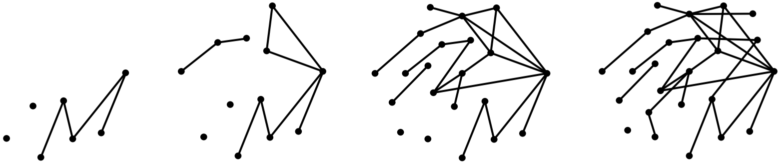
Given essentially countable directed category with amalgamation $\mathcal{K}$ there exists a unique Fraisse sequence:



In this presentation we may assume that $A_n \subset A_{n+1}$ and think of the graph $G = \bigcup_n A_n$.
Kubiś proved some results about uniqueness and homogeneity of such sequences.

Why Fraisse limits?

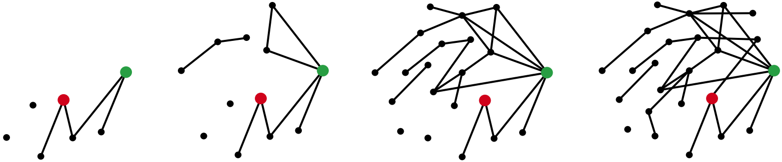
Given Fraisse sequence (G_n), $G = \bigcup G_n$



and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.
Then hope that (H_n) is again a Fraisse sequence.
Moreover, it follows from Kubiś that whenever $i : V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

Why Fraisse limits?

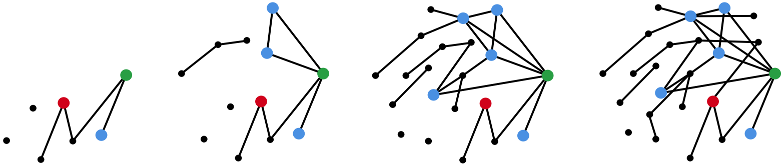
Given Fraisse sequence (G_n), $G = \bigcup G_n$



and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.
Then hope that (H_n) is again a Fraisse sequence.
Moreover, it follows from Kubiś that whenever $i\colon V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

Why Fraisse limits?

Given Fraisse sequence (G_n), $G = \bigcup G_n$



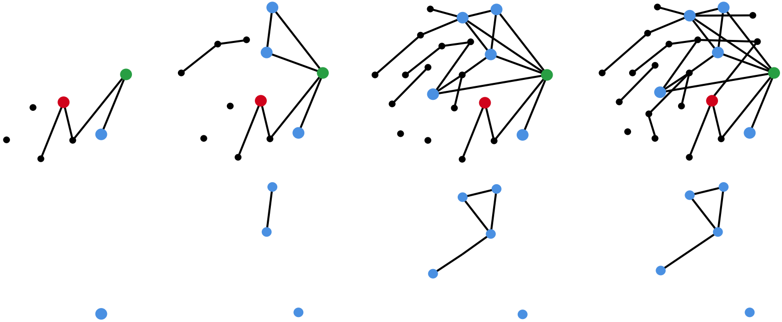
and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.

Then hope that (H_n) is again a Fraisse sequence.

Moreover, it follows from Kubiś that whenever $i\colon V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

Why Fraisse limits?

Given Fraisse sequence (G_n), $G = \bigcup G_n$

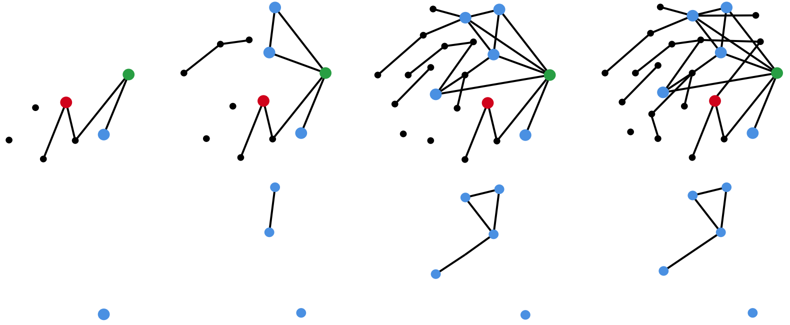


and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.
Then hope that (H_n) is again a Fraisse sequence.

Moreover, it follows from Kubiś that whenever $i \colon V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

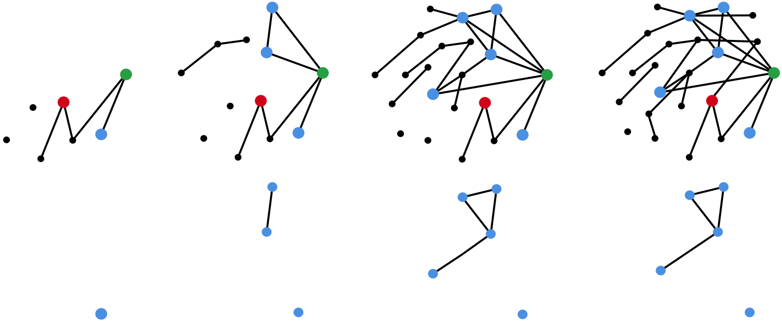
Why Fraisse limits?

Given Fraisse sequence (G_n), $G = \bigcup G_n$



and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.
Then hope that (H_n) is again a Fraisse sequence.
Moreover, it follows from Kubiś that whenever $i\colon V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

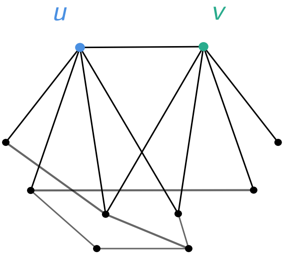
Given Fraisse sequence (G_n), $G = \bigcup G_n$



and $A, B \subset G$, we may define $H_n := G_n \cap \bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$.
Then hope that (H_n) is again a Fraisse sequence.
Moreover, it follows from Kubiś that whenever $i\colon V \to V'$ is isomorphism between finite subgraphs of G, and there are A, n such that $V \subset A$, $V' \subset G_n$, and i may be extended to an arrow from A to G_n, then i extend to an automorphism of G.

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

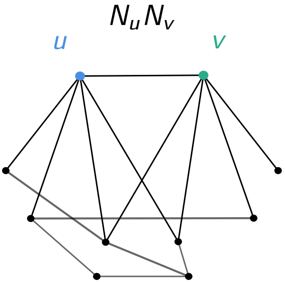


It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Properties of NN^c

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.



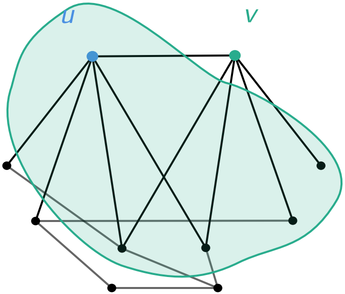
It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Properties of NN^c

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

$N_u N_v$

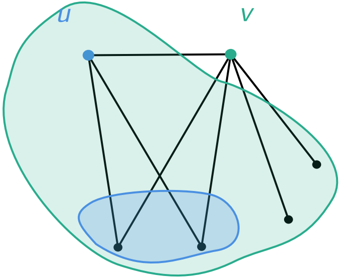


It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

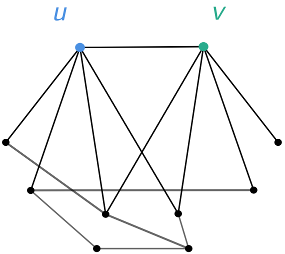
$N_u N_v$



It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Observation

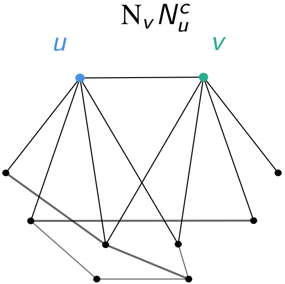
Discuss 1-NN^c graph G and $v, w \in G$, vEw.



It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

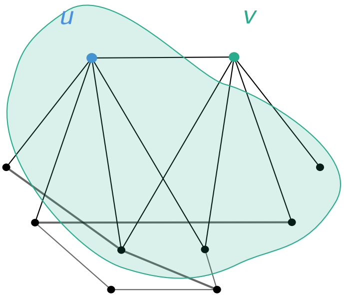


It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

$\mathrm{N}_v N_u^c$



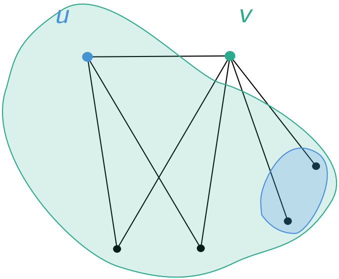
It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Properties of NN^c

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

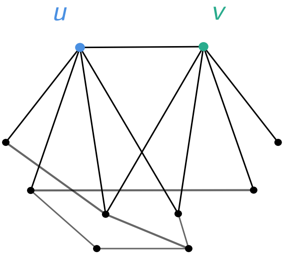
$N_v N_u^c$



It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

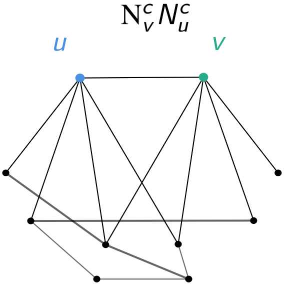


It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Properties of NN^c

Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.

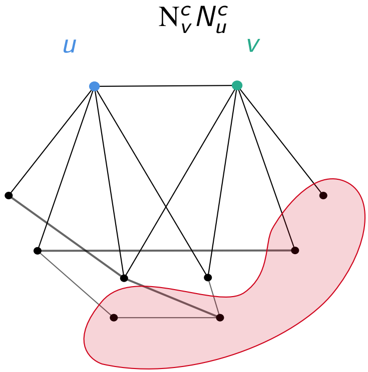


It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Properties of NN^c

Observation

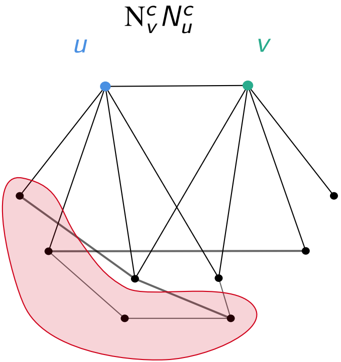
Discuss 1-NN^c graph G and $v, w \in G$, vEw.



It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

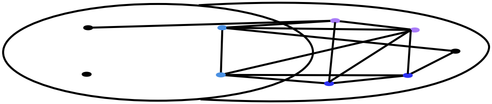
Observation

Discuss 1-NN^c graph G and $v, w \in G$, vEw.



It has analogoues for higher k. eg. 2-NN^c implies that for every clique v_1, v_2, v_3 we have $N_{v_1} N_{v_2} N_{v_3} \sim G$.

Generalization



$k \geq 2$, language E, P- relational symbols, E binary, and P of arity k.

For $\{u_0, \ldots, u_{k-1}\}, \{v_0, \ldots, v_{k-1}\} \in P(G)$

$$\{u_0, \ldots, u_{k-1}\} R \{v_0, \ldots, v_{k-1}\} \text{ iff } \bigwedge_{j=0}^{k-1} \neg u_i E v_j \text{ for some } i \leq k-1.$$

Finite structure (G, E, P) is called a *k-perturbed graph* if

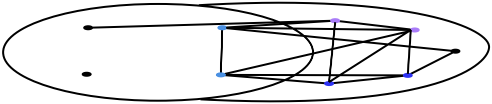
(A) E is a graph relation;

(B) P is symmetric;

(C) Different pertubed k-sets are disjoint cliques;

(D) transitive relation $\prec$ generated by R is a partial ordering.

Generalization



$k \geq 2$, language E, P- relational symbols, E binary, and P of arity k.
For $\{u_0, \ldots, u_{k-1}\}, \{v_0, \ldots, v_{k-1}\} \in P(G)$

$$\{u_0, \ldots, u_{k-1}\} R \{v_0, \ldots, v_{k-1}\} \text{ iff } \bigwedge_{j=0}^{k-1} \neg u_i E v_j \text{ for some } i \leq k-1.$$

Finite structure (G, E, P) is called a *k-perturbed graph* if

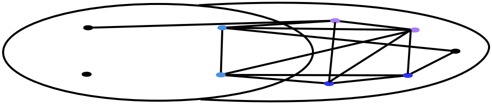
(A) E is a graph relation;

(B) P is symmetric;

(C) Different pertubed k-sets are disjoint cliques;

(D) transitive relation $\prec$ generated by R is a partial ordering.

Generalization



$k \geq 2$, language E, P- relational symbols, E binary, and P of arity k.
For $\{u_0, \ldots, u_{k-1}\}, \{v_0, \ldots, v_{k-1}\} \in P(G)$

$$\{u_0, \ldots, u_{k-1}\} R \{v_0, \ldots, v_{k-1}\} \text{ iff } \bigwedge_{j=0}^{k-1} \neg u_i E v_j \text{ for some } i \leq k-1.$$

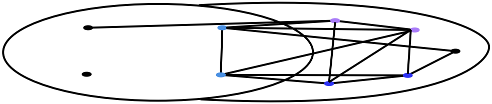
Finite structure (G, E, P) is called a *k-perturbed graph* if

(A) E is a graph relation;

(B) P is symmetric;

(C) Different pertubed k-sets are disjoint cliques;

(D) transitive relation $\prec$ generated by R is a partial ordering.



$k \geq 2$, language E, P- relational symbols, E binary, and P of arity k. For $\{u_0, \dots, u_{k-1}\}, \{v_0, \dots, v_{k-1}\} \in P(G)$

$$\{u_0, \dots, u_{k-1}\} R \{v_0, \dots, v_{k-1}\} \text{ iff } \bigwedge_{j=0}^{k-1} \neg u_i E v_j \text{ for some } i \leq k-1.$$

Finite structure (G, E, P) is called a *k-perturbed graph* if

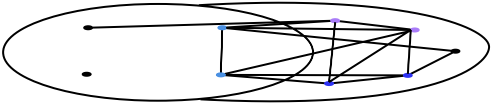
(A) E is a graph relation;

(B) P is symmetric;

(C) Different pertubed k-sets are disjoint cliques;

(D) transitive relation $\prec$ generated by R is a partial ordering.

Generalization



$k \geq 2$, language E, P- relational symbols, E binary, and P of arity k.
For $\{u_0, \ldots, u_{k-1}\}, \{v_0, \ldots, v_{k-1}\} \in P(G)$

$$\{u_0, \ldots, u_{k-1}\} R \{v_0, \ldots, v_{k-1}\} \text{ iff } \bigwedge_{j=0}^{k-1} \neg u_i E v_j \text{ for some } i \leq k-1.$$

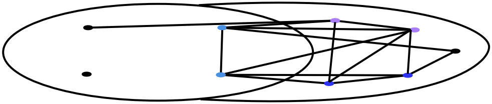
Finite structure (G, E, P) is called a *k-perturbed graph* if

(A) E is a graph relation;

(B) P is symmetric;

(C) Different pertubed k-sets are disjoint cliques;

(D) transitive relation $\prec$ generated by R is a partial ordering.



k-perturbed $(G_1, E_1, P_1), (G_2, E_2, P_2)$, $f : G_1 \to G_2$ is *k-embedding*, provided that

(a) $uE_1v \implies f(u)E_2f(v)$, i.e. f is a graph embedding;

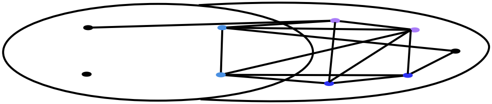
(b) $P_1(u_0, \ldots, u_{k-1}) \implies P_2(f(u_0), \ldots, f(u_{k-1}))$,
i.e. f maps perturbed k-sets in G_1 onto perturbed k-sets in G_2;

(c) If $\bar{u} \in P_2 \setminus \{\{f(\bar{v})\} : \bar{v} \in P_1\}$, then $\bar{u} \subseteq G_2 \setminus f[G_1]$,
i.e. f preserves non-pertubed elements.

(d) Given $\{u_0, \ldots, u_{k-1}\} \in P(G_1)$ and $w \in G_2 \setminus f[G_1]$, we have $wE_2f(u_i)$, for at least one $i < k$,
i.e. non-neighborhoods of perturbed k-sets in G_1 is preserved by f - they cannot be enlarged;

By $\mathscr{PG}_k$ we denote the category of k-perturbed finite graphs with k-embeddings.

Generalization



k-perturbed $(G_1, E_1, P_1), (G_2, E_2, P_2)$, $f : G_1 \to G_2$ is *k-embedding*, provided that

(a) $uE_1v \implies f(u)E_2f(v)$, i.e. f is a graph embedding;

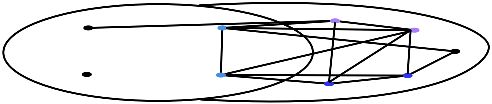
(b) $P_1(u_0, \ldots, u_{k-1}) \implies P_2(f(u_0), \ldots, f(u_{k-1}))$,
i.e. f maps perturbed k-sets in G_1 onto perturbed k-sets in G_2;

(c) If $\bar{u} \in P_2 \setminus \{\{f(\bar{v})\} : \bar{v} \in P_1\}$, then $\bar{u} \subseteq G_2 \setminus f[G_1]$,
i.e. f preserves non-pertubed elements.

(d) Given $\{u_0, \ldots, u_{k-1}\} \in P(G_1)$ and $w \in G_2 \setminus f[G_1]$, we have $wE_2f(u_i)$, for at least one $i < k$,
i.e. non-neighborhoods of perturbed k-sets in G_1 is preserved by f - they cannot be enlarged;

By $\mathscr{PG}_k$ we denote the category of k-perturbed finite graphs with k-embeddings.

Generalization



k-perturbed $(G_1, E_1, P_1), (G_2, E_2, P_2)$, $f: G_1 \to G_2$ is *k-embedding*, provided that

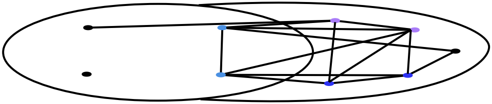
(a) $uE_1v \implies f(u)E_2f(v)$, i.e. f is a graph embedding;

(b) $P_1(u_0, \ldots, u_{k-1}) \implies P_2(f(u_0), \ldots, f(u_{k-1}))$,
i.e. f maps perturbed k-sets in G_1 onto perturbed k-sets in G_2;

(c) If $\bar{u} \in P_2 \setminus \{\{f(\bar{v})\} : \bar{v} \in P_1\}$, then $\bar{u} \subseteq G_2 \setminus f[G_1]$,
i.e. f preserves non-pertubed elements.

(d) Given $\{u_0, \ldots, u_{k-1}\} \in P(G_1)$ and $w \in G_2 \setminus f[G_1]$, we have $wE_2f(u_i)$, for at least one $i < k$,
i.e. non-neighborhoods of perturbed k-sets in G_1 is preserved by f - they cannot be enlarged;

By $\mathscr{PG}_k$ we denote the category of k-perturbed finite graphs with k-embeddings.



k-perturbed $(G_1, E_1, P_1), (G_2, E_2, P_2)$, $f\colon G_1 \to G_2$ is *k-embedding*, provided that

(a) $uE_1v \implies f(u)E_2f(v)$, i.e. f is a graph embedding;

(b) $P_1(u_0, \ldots, u_{k-1}) \implies P_2(f(u_0), \ldots, f(u_{k-1}))$, i.e. f maps perturbed k-sets in G_1 onto perturbed k-sets in G_2;

(c) If $\bar{u} \in P_2 \setminus \{\{f(\bar{v})\} : \bar{v} \in P_1\}$, then $\bar{u} \subseteq G_2 \setminus f[G_1]$, i.e. f preserves non-pertubed elements.

(d) Given $\{u_0, \ldots, u_{k-1}\} \in P(G_1)$ and $w \in G_2 \setminus f[G_1]$, we have $wE_2f(u_i)$, for at least one $i < k$, i.e. non-neighborhoods of perturbed k-sets in G_1 is preserved by f - they cannot be enlarged;

By $\mathscr{PG}_k$ we denote the category of k-perturbed finite graphs with k-embeddings.

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation

Explain how to get NN^c.

Family of types

$G \in \mathscr{PG}_k$, $\overline{u} \in P(G)$,

initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v} \in P(G),\ \overline{v} \preccurlyeq \overline{u}} \ \bigcap_{v \in \overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.

By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple

Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation
Explain how to get NN^c.
Family of types
$G \in \mathscr{PG}_k$, $\overline{u} \in P(G)$,
initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v} \in P(G),\ \overline{v} \preccurlyeq \overline{u}} \ \bigcap_{v \in \overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.
By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple
Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation
Explain how to get NN^c.
Family of types
$G \in \mathscr{PG}_k, \overline{u} \in P(G)$,
initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v} \in P(G),\ \overline{v} \preccurlyeq \overline{u}} \ \bigcap_{v \in \overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.
By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple
Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation
Explain how to get NN^c.
Family of types
$G \in \mathscr{PG}_k, \overline{u} \in P(G)$,
initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v} \in P(G),\ \overline{v} \preccurlyeq \overline{u}} \ \bigcap_{v \in \overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.
By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple
Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation
Explain how to get NN^c.
Family of types
$G \in \mathscr{PG}_k, \overline{u} \in P(G)$,
initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v} \in P(G),\ \overline{v} \preccurlyeq \overline{u}} \ \bigcap_{v \in \overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.
By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple
Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Putting the pieces together

Check that $\mathscr{PG}_k$ has amalgamation
Explain how to get NN^c.
Family of types
$G \in \mathscr{PG}_k$, $\overline{u} \in P(G)$,
initial segment of $\overline{u}$ is

$$\bigcup_{\overline{v}\in P(G),\ \overline{v}\preccurlyeq\overline{u}} \ \bigcap_{v\in\overline{v}} N_v^c.$$

$H \subset G$ closure of H is the union of all initial segments of those elements of $P(G)$ which intersects H.
By type we mean any graph $G \in \mathscr{PG}_k$ equal to the closure of some k-tuple
Relativize construction to any downward-closed family of k-types $\mathscr{T}$

Evolution from the single point

G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \ldots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Non-universal NN^c graphs

Evolution from the single point
G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \ldots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Non-universal NN^c graphs

Evolution from the single point
G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \ldots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Non-universal NN^c graphs

Evolution from the single point
G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \dots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Non-universal NN^c graphs

Evolution from the single point
G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \ldots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Non-universal NN^c graphs

Evolution from the single point
G finite graph of cardinality n.
$G' = G \cup \{\hat{g}\}$ is immediate descendant of G, if $\hat{g} \notin G$, and there exists an injective sequence $\overline{g} \in G^n$, $\overline{i} \in 2^n$, such that if we define:

- $A_l = \bigcap_{j=0}^{l} N_{g_j}^{i_j}$, where N_v^0 stands for non-neigbourhood of v;
- $k = \min\{j < n-1 \colon g_{j+1} \notin A_j\} \cup \{n-1\}$;

then $\hat{g} E g_j$ iff $i_j = 1$ and $j \leq k$.
Arrow: graph embedding $f \colon G \to H$, if there exists $G_0, G_1, \ldots G_n$ such that $G_0 = f[G]$, $G_n = H$ and G_{i+1} is immediate descendant of G_i
Generalization with allowed family of added points.

Extending partial automorphism

Fix G- one of graphs produced on previous slides, finite $A, B \subset G$ and graph isomorphism $i\colon A \to B$ i may not extend to the automorphism of the whole G. Example!

i may be extended to the automorphism of G iff i extends to the closure of A and B (essentially for free!)

Extending partial automorphism

Fix G- one of graphs produced on previous slides, finite $A, B \subset G$ and graph isomorphism $i\colon A \to B$ i may not extend to the automorphism of the whole G. Example!

i may be extended to the automorphism of G iff i extends to the closure of A and B (essentially for free!)

Extending partial automorphism

Fix G- one of graphs produced on previous slides, finite $A, B \subset G$ and graph isomorphism $i: A \to B$ i may not extend to the automorphism of the whole G. Example!
i may be extended to the automorphism of G iff i extends to the closure of A and B (essentially for free!)

Extending partial automorphism

Fix G- one of graphs produced on previous slides, finite $A, B \subset G$ and graph isomorphism $i\colon A \to B$ i may not extend to the automorphism of the whole G. Example!
i may be extended to the automorphism of G iff i extends to the closure of A and B (essentially for free!)

Summary

For every k, we got $2 \cdot 2 \cdot \omega \cdot \mathfrak{c}$ many pairwise non-isomorphic k-NN^c graphs which are not $k+1$-NN^c.

Moreover, full characterization of extendable finite automorphism seems to just work.

Summary

For every k, we got $2 \cdot 2 \cdot \omega \cdot \mathfrak{c}$ many pairwise non-isomorphic k-NN^c graphs which are not $k+1$-NN^c.
Moreover, full characterization of extendable finite automorphism seems to just work.

Open problems

Question

Is there a rigit k-NN^c graph for some k?

Question

Suppose that G is k-NN^c and any disjoint $A, B \subset G$ with $\mathrm{card}(A \cup B) \leq k+1$ set $\bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N^c_w$ is infinite, and this property is hereditary. Is G necessarily $(k+1)$-NN^c?

Open problems

Question

Is there a rigit k-NN^c graph for some k?

Question

Suppose that G is k-NN^c and any disjoint $A, B \subset G$ with $\mathrm{card}(A \cup B) \leq k+1$ set $\bigcap_{v \in A} N_v \cap \bigcap_{w \in B} N_w^c$ is infinite, and this property is hereditary. Is G necessarily $(k+1)$-NN^c?

Bibliography

Sz. Głąb, A. Widz, J. Swaczyna, On graph's answering Bonato's question, in preparation

P. Gordinowicz, On graphs isomorphic to their neighbour and non-neighbour sets, European J. Combin. 31 (2010), no. 5, 1419–1428. MR 2644430

G. Andrzejczak, P. Gordinowicz, On automorphisms of the countable p.e.c. graph, Discrete Math. 313 (2013), no. 11, 1206–1211. MR 3034751

P. Cameron, The random graph, Algorithms and Combinatorics 14 (2013).

W. Kubiś, Fraısse sequences: category-theoretic approach to universal homogeneous structures, Ann. Pure Appl. Logic 165 (2014), no. 11, 1755–1811. MR 3244668

Thank you for your attention! Děkuji za pozornost!
Grazie per l'attenzione! Köszönöm a figyelmet!
Dziękuję za uwagę!
Gracias por su atención!
$\Sigma\alpha\zeta\ \varepsilon\upsilon\chi\alpha\iota\sigma\tau\,\omega\ \gamma\iota\alpha\tau\eta\nu\ \pi\rho\sigma\chi\,\eta\ \sigma\alpha$!
Gratias pro vobis animus attentus!
Danke für Ihre Aufmerksamkeit!
Bedankt voor uw aandacht! Kiitos huomiostasi!
ご清ありがとうございました！
Ďakujem za vašu pozornosť!