Slaloms, cardinal invariants, and selection principles

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Cardona M.A., Gavalová V., Mejía D.A., Repický M., and Šupina J., Slalom numbers, submitted, arXiv:2406.19901.

Slaloms historically

TRANSACTIONS OF THE
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Volume 281, Number 1, January 1984

ADDITIVITY OF MEASURE IMPLIES ADDITIVITY OF CATEGORY

BY

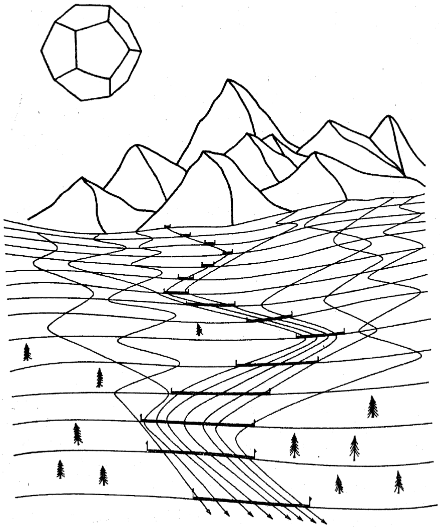
TOMEK BARTOSZYŃSKI

https://www.ams.org/journals/tran/1984-281-01/
S0002-9947-1984-0719666-7/

A (standard) slalom

A sequence of finite sets with cardinality of the n-th term being at most n.

$$\boxed{\prod_{n\in\omega}[\omega]^{\leq n}}$$



http://matwbn.icm.edu.pl/ksiazki/fm/fm127/fm127118.pdf

$$\min\left\{|S| : S \subseteq \prod_{n\in\omega}[\omega]^{\leq n},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^{\infty} s\right\}$$

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- notation $\mathfrak{sl}_{\mathrm{e}}$

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- relational system $\mathfrak{d}({}^{\omega}\omega, \prod_{n\in\omega}[\omega]^{\leq n}, \in^{\infty})$

Selection principles

$$\boxed{\mathrm{S}_1(\Gamma,\Gamma)}$$

$\boxed{\mathrm{S}_1(\Gamma, \Gamma)}$

Definition (M. Scheepers 1996)

A topological space X satisfies the selection principle $\mathrm{S}_1(\Gamma, \Gamma)$ if for each sequence $\langle\langle V_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ with $V_{n,m}$ being open subsets of X such that

$$(\forall n \in \omega)\, x \in^* V_{n,m}$$

for each $x \in X$, there is a $d \in {}^{\omega}\omega$ with $x \in^* V_{n,d(n)}$ for each $x \in X$.

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Theorem

(a) *Any Hausdorff $\mathrm{S}_1(\Gamma_{\mathrm{id}}, \Gamma)$-space is totally imperfect, i.e., does not contain any perfect set.*

(b) *No uncountable Polish space is $\mathrm{S}_1(\Gamma_{\mathrm{id}}, \Gamma)$.*

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Theorem

$\mathtt{non}(\mathrm{S}_1(\Gamma_{\mathrm{id}}, \Gamma)) = \mathfrak{sl}_{\mathrm{e}}$.

$\boxed{\mathrm{S}_1(\Gamma, \Gamma)}$

- $\mathtt{non}(\mathrm{S}_1(\Gamma, \Gamma)) = \mathfrak{b}$

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- $\mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{S}_1(\Gamma_{\mathrm{id}},\Gamma)$

$\mathrm{S}_1(\Gamma, \Gamma)$

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- $\mathrm{S}_1(\Gamma, \Gamma) \to \mathrm{S}_1(\Gamma_{\mathrm{id}}, \Gamma)$
- If $\mathfrak{b} < \mathtt{non}(\mathcal{M})$ then $\mathrm{S}_1(\Gamma, \Gamma) \not\equiv \mathrm{S}_1(\Gamma_{\mathrm{id}}, \Gamma)$.

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- $\mathtt{non}(S_1(\Gamma, \Gamma)) = \mathfrak{b}$
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Question

If $\mathfrak{b} = \mathtt{non}(\mathcal{M})$*, is it true that* $S_1(\Gamma, \Gamma) \not\equiv S_1(\Gamma_{\mathrm{id}}, \Gamma)$*?*

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- Any $\mathrm{S}_1(\Gamma,\Gamma)$-set of reals is perfectly meager, i.e., for any perfect set $P \subseteq \mathbb{R}$, the intersection $A \cap P$ is meager in the subspace P.

$\boxed{\mathrm{S}_1(\Gamma,\Gamma)}$

- $\mathtt{non}(\mathrm{S}_1(\Gamma,\Gamma)) = \mathfrak{b}$
- $\mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{S}_1(\Gamma_{\mathrm{id}},\Gamma)$
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Question

If $\mathfrak{b} = \mathtt{non}(\mathcal{M})$, is it true that $\mathrm{S}_1(\Gamma,\Gamma) \not\equiv \mathrm{S}_1(\Gamma_{\mathrm{id}},\Gamma)$?

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Question

Is any $\mathrm{S}_1(\Gamma_{\mathrm{id}},\Gamma)$-set of reals perfectly meager?

Slaloms

A slalom

A sequence of sets of natural numbers.

$$\min \left\{|S| : S \subseteq \prod_{n\in\omega}[\omega]^{\leq n},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^{\infty} s\right\}$$

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$$\min\{|S| : S \subseteq {}^{\omega}\mathtt{nwd},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^{\infty} s\}$$

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Theorem (M. Canjar 1988)

In $V^{\mathbb{C}_\lambda}$, if $\mu \leq \lambda$ is regular, then there is an ultrafilter $\mathcal{U}_\mu$ such that $\mathfrak{b}_{\mathcal{U}_\mu} = \mu$.

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Corollary

In $V^{\mathbb{C}_\lambda}$, if $\mu_1 < \mu_2 \le \lambda$ are regular, then there are ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ such that

$$\mathrm{S}_1(\Gamma, \mathcal{U}_1\text{-}\Gamma) \not\equiv \mathrm{S}_1(\Gamma, \mathcal{U}_2\text{-}\Gamma).$$

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Question

If $\mathfrak{b} = \mathfrak{d}$, is it true that $\mathrm{S}_1(\Gamma, J_1\text{-}\Gamma) \not\equiv \mathrm{S}_1(\Gamma, J_2\text{-}\Gamma)$ for some J_1, J_2?

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- notation $\mathfrak{sl}_{\mathrm{t}}$
- other notation $\mathfrak{d}^{\mathrm{Lc}}_{\omega,\mathrm{id}}, \mathfrak{d}_{\mathrm{id}}(\in^*), \mathfrak{c}^{\forall}_{\omega,\mathrm{id}}$
- is equal to $\mathtt{cof}(\mathcal{N})$ (T. Bartoszyński 1984)

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Theorem

$\mathbf{non}(\mathrm{S}_1(\Gamma_{\mathrm{id}}, \mathcal{O})) = \mathfrak{sl}_{\mathrm{t}}$.

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Theorem

$\mathtt{non}(\mathrm{S}_1(\Gamma_{\mathrm{id}}, \mathcal{O})) = \mathfrak{sl}_{\mathrm{t}}$.

$\min\{\lvert S\rvert : S \subseteq {}^{\omega}\mathrm{Fin},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^* s\}$	$=$	$\mathfrak{d}$	$\mathrm{S}_1(\Gamma, \mathcal{O})$
$\min\{\lvert S\rvert : S \subseteq \mathtt{FUPC},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^* s\}$	$=$	$\mathtt{cov}(\mathcal{M})$	$\mathrm{S}_1(\mathcal{O}, \mathcal{O})$
$\min\left\{\lvert S\rvert : S \subseteq {}^{\omega}\mathrm{Fin},\ (\forall x \in {}^{\omega}\omega)(\exists s \in S)\ x \in^{J^d} s\right\}$	$=$	$\mathfrak{d}_J$	$\mathrm{S}_1(\Gamma, J\text{-}\Lambda)$

General framework

$$\mathfrak{sl}_{\mathrm{e}}(E,J) = \mathfrak{d}({}^{\omega}\omega, E, \in^{J^+}) \text{ and } \mathfrak{sl}_{\mathrm{t}}(E,J) = \mathfrak{d}({}^{\omega}\omega, E, \in^{J^d})$$

$$\mathfrak{sl}_{\mathrm{e}}(E,J) = \mathfrak{d}({}^{\omega}\omega, E, \in^{J^+}) \text{ and } \mathfrak{sl}_{\mathrm{t}}(E,J) = \mathfrak{d}({}^{\omega}\omega, E, \in^{J^d})$$

Theorem

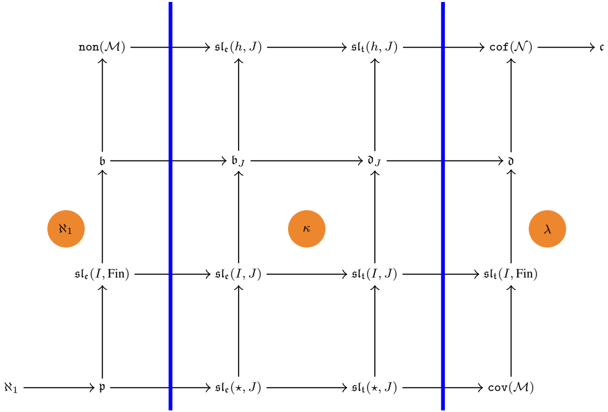
If E is a family of functions with domain ω then

$$\begin{aligned} \mathfrak{sl}_{\mathrm{e}}(E,J) &= \mathtt{non}(\mathrm{S}_1(E\text{-}\Gamma, J\text{-}\Gamma)), \\ \mathfrak{sl}_{\mathrm{t}}(E,J) &= \mathtt{non}(\mathrm{S}_1(E\text{-}\Gamma, (J^d)^c\text{-}\Gamma)). \end{aligned}$$

$$
\begin{array}{ccccccccccc}
 & & \mathtt{non}(\mathcal{M}) & \longrightarrow & \mathfrak{sl}_{\mathfrak{e}}(h,J) & \longrightarrow & \mathfrak{sl}_{\mathfrak{t}}(h,J) & \longrightarrow & \mathtt{cof}(\mathcal{N}) & \longrightarrow & \mathfrak{c} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathfrak{b} & \longrightarrow & \mathfrak{b}_J & \longrightarrow & \mathfrak{d}_J & \longrightarrow & \mathfrak{d} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathfrak{sl}_{\mathfrak{e}}(I,\mathrm{Fin}) & \longrightarrow & \mathfrak{sl}_{\mathfrak{e}}(I,J) & \longrightarrow & \mathfrak{sl}_{\mathfrak{t}}(I,J) & \longrightarrow & \mathfrak{sl}_{\mathfrak{t}}(I,\mathrm{Fin}) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\aleph_1 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{sl}_{\mathfrak{e}}(\star,J) & \longrightarrow & \mathfrak{sl}_{\mathfrak{t}}(\star,J) & \longrightarrow & \mathtt{cov}(\mathcal{M}) & &
\end{array}
$$

Theorem

Let $\lambda = \lambda^{\aleph_0}$ be an infinite cardinal. Then, in $V^{\mathbb{C}_\lambda}$, any uncountable regular cardinal κ satisfying $\lambda^{<\kappa} = \lambda$ is a slalom number of the form $\mathfrak{sl}_{\mathrm{e}}(\star, J) = \mathfrak{sl}_{\mathrm{t}}(h, J)$ for some maximal ideal J on ω.



Disjoint sum of ideals

Theorem

Let I_0, I_1, J_0 and J_1 be ideals on ω. Then:

(a) $\mathfrak{d}_{J_0 \oplus J_1} = \max\{\mathfrak{d}_{J_0}, \mathfrak{d}_{J_1}\}$.

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(c) $\mathfrak{sl}_{\mathrm{t}}(I_0, J_0 \oplus J_1) = \max\{\mathfrak{sl}_{\mathrm{t}}(I_0, J_0), \mathfrak{sl}_{\mathrm{t}}(I_0, J_1)\}$.

(d) $\mathfrak{sl}_{\mathrm{e}}(I_0, J_0 \oplus J_1) = \min\{\mathfrak{sl}_{\mathrm{e}}(I_0, J_0), \mathfrak{sl}_{\mathrm{e}}(I_0, J_1)\}$.

(e) $\mathfrak{sl}_{\mathrm{t}}(\star, J_0 \oplus J_1) = \max\{\mathfrak{sl}_{\mathrm{t}}(\star, J_0), \mathfrak{sl}_{\mathrm{t}}(\star, J_1)\}$.

(f) $\mathfrak{sl}_{\mathrm{e}}(\star, J_0 \oplus J_1) = \min\{\mathfrak{sl}_{\mathrm{e}}(\star, J_0), \mathfrak{sl}_{\mathrm{e}}(\star, J_1)\}$.

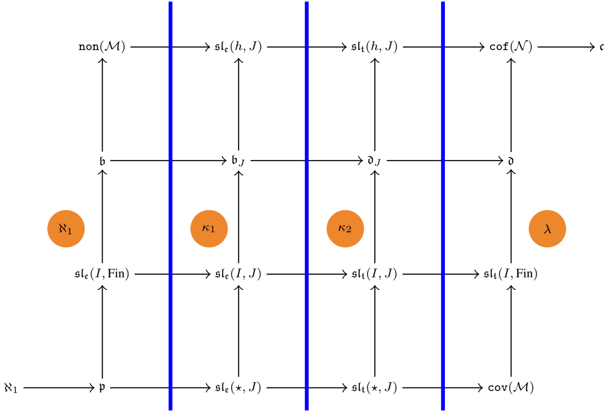
(g) $\mathfrak{sl}_{\mathrm{t}}(h_0 \oplus h_1, J_0 \oplus J_1) = \max\{\mathfrak{sl}_{\mathrm{t}}(h_0, J_0), \mathfrak{sl}_{\mathrm{t}}(h_1, J_1)\}$.

(h) $\mathfrak{sl}_{\mathrm{e}}(h_0 \oplus h_1, J_0 \oplus J_1) = \min\{\mathfrak{sl}_{\mathrm{e}}(h_0, J_0), \mathfrak{sl}_{\mathrm{e}}(h_1, J_1)\}$.

Theorem

Let $\lambda = \lambda^{\aleph_0}$ be an infinite cardinal. Then, in $V^{\mathbb{C}_\lambda}$, for any regular $\aleph_1 \leq \kappa_1 \leq \kappa_2$, if $\lambda^{<\kappa_2} = \lambda$ then there is some ideal J on ω such that

$$\mathfrak{sl}_\mathrm{e}(\star, J) = \mathfrak{sl}_\mathrm{e}(h, J) = \kappa_1 \text{ and } \mathfrak{sl}_\mathrm{t}(\star, J) = \mathfrak{sl}_\mathrm{t}(h, J) = \kappa_2.$$



Lemma (M. Canjar 1988)

Let $J \subseteq \mathcal{P}(\omega)$ *be an ideal. If* $c \in {}^{\omega}\omega$ *is Cohen over* V*, then* $x \leq^{J'} c$ *for all* $x \in {}^{\omega}\omega \cap V$*,*

$$J' = \langle J \cup \{\{i < \omega : \, c(i) < x(i)\} : \, x \in {}^{\omega}\omega \cap V\}\rangle.$$

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Question

Is there any model where all four rows of the diagram are different for some pair I, J?

Baire property

Theorem

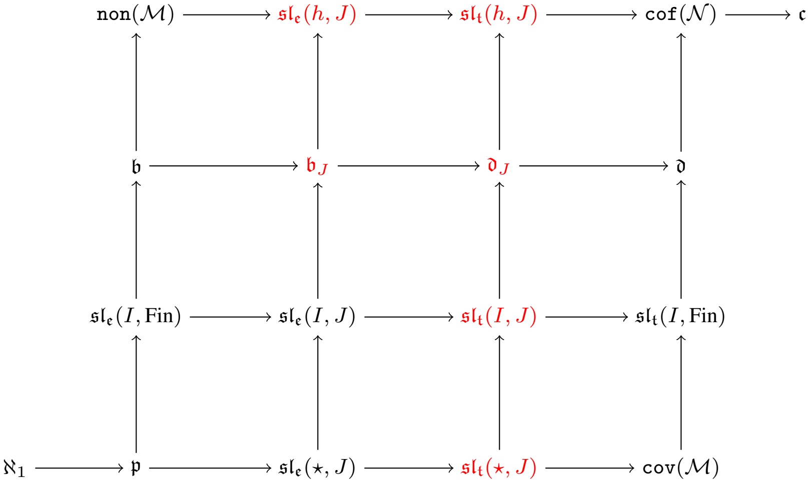
Let I and J be ideals on ω, J with the Baire property, and $h \in {}^{\omega}\omega$. Then

(a) $\mathfrak{sl}_{\mathrm{e}}(h, J) = \mathtt{non}(\mathcal{M})$.

(b) $\mathfrak{sl}_{\mathrm{t}}(h, J) = \mathtt{cof}(\mathcal{N})$.

(c) $\mathfrak{sl}_{\mathrm{t}}(I, J) = \mathfrak{sl}_{\mathrm{t}}(I, \mathrm{Fin})$.

(d) $\mathfrak{sl}_{\mathrm{t}}(\star, J) = \mathtt{cov}(\mathcal{M})$.



$\mathrm{non}(\mathcal{M})$
$\mathfrak{sl}_\mathfrak{e}(h, J)$
$\mathfrak{sl}_\mathfrak{t}(h, J)$
$\mathrm{cof}(\mathcal{N})$
$\mathfrak{c}$
$\mathfrak{b}$
$\mathfrak{b}_J$
$\mathfrak{d}_J$
$\mathfrak{d}$
$\mathfrak{sl}_\mathfrak{e}(I, \mathrm{Fin})$
$\mathfrak{sl}_\mathfrak{e}(I, J)$
$\mathfrak{sl}_\mathfrak{t}(I, J)$
$\mathfrak{sl}_\mathfrak{t}(I, \mathrm{Fin})$
$\aleph_1$
$\mathfrak{p}$
$\mathfrak{sl}_\mathfrak{e}(\star, J)$
$\mathfrak{sl}_\mathfrak{t}(\star, J)$
$\mathrm{cov}(\mathcal{M})$

Question

Do we have that $\mathfrak{sl}_{\mathrm{e}}(I, J) = \mathfrak{sl}_{\mathrm{e}}(I, \mathrm{Fin})$ *and* $\mathfrak{sl}_{\mathrm{e}}(\star, J) = \mathfrak{sl}_{\mathrm{e}}(\star, \mathrm{Fin})$ *when* J *has the Baire property?*

Question

Is $\mathrm{S}_1(\Gamma_h, J\text{-}\Gamma)$ *equivalent to* $\mathrm{S}_1(\Gamma_h, \Gamma)$ *when* J *has the Baire property? The same applies to* $\mathrm{S}_1(I\text{-}\Gamma, J\text{-}\Lambda)$, $\mathrm{S}_1(\Omega, J\text{-}\Lambda)$, *and* $\mathrm{S}_1(\Gamma, J\text{-}\Gamma)$.

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Thanks for your attention!

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