Separating Regularity Properties with the Raisonnier Filter

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Lebesgue Measure and Inaccessibles

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"Can you take Solovay's Inaccessible Away?"

'Yes' for the BP but 'No' for LM. More specifically:

$$\mathbf{\Sigma}^1_3(\mathsf{LM}) \;\Rightarrow\; \forall a \in \mathbb{R}(\aleph_1^{L[a]} < \aleph_1)$$

From the last statement, it is not hard to deduce that $L \models (\aleph_1^V$ is inaccessible)

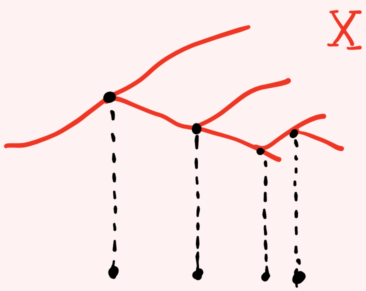
The Raisonnier Filter

This proof was simplified using a purely combinatorial argument.

Definition (Raisonnier 1984)

- For $x, y \in 2^\omega$, let $h(x,y) := \min\{n \mid x{\restriction}n \neq y{\restriction}n\}$.
  ("breaking point" of x, y)

- For $X \subseteq 2^\omega$, let $H(X) := \{h(x,y) \mid x, y \in X\} \subseteq \omega$.
  (set of "all possible break-points" for reals from X)



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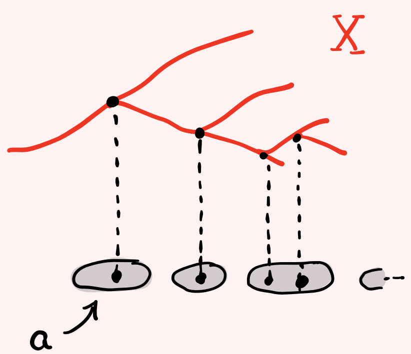

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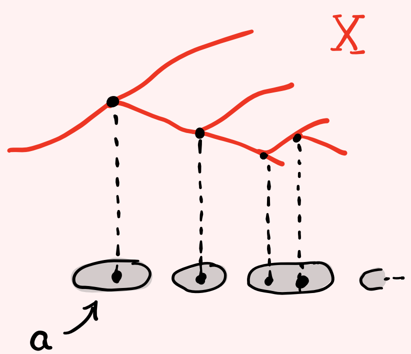

- For $X \subseteq 2^\omega$, let $H(X) := \{h(x,y) \mid x, y \in X\} \subseteq \omega$.
  (set of "all possible break-points" for reals from X)
- If $a \subseteq \omega$ and $X \subseteq 2^\omega$, then we say that X breaks in a if $H(X) \subseteq a$.
- Let M be a model of set theory. Then $\mathcal{F}_M$ is the collection of all $a \subseteq \omega$ such that the reals of M can be covered by countably many sets which break in a, i.e.,

$$\exists X_0, X_1, \dots \subseteq 2^\omega \; (2^\omega \cap M \subseteq \bigcup_n X_n \text{ and } H(X_n) \subseteq a)$$

The Raisonnier Filter

Theorem (Raisonnier 1984)

- If the reals of M have size $\aleph_1$ then $\mathcal{F}_M$ is a non-trivial filter on ω.
- If $2^\omega \cap M$ is a $\mathbf{\Sigma}^1_n$ set, then $\mathcal{F}_M$ is $\mathbf{\Sigma}^1_{n+1}$.

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Theorem (Raisonnier-Shelah 1984)

Assume that $\aleph_1^{L[a]} = \aleph_1$ and all $\mathbf{\Sigma}^1_2$ sets are Lebesgue-measurable.

Then the Raisonnier Filter $\mathcal{F}_{L[a]}$ is a rapid filter: This means that the collection of increasing enumerations of sets in $\mathcal{F}_{L[a]}$ forms a dominating family in ω^ω.

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Theorem (Mokobodzki 1967)

A rapid filter is not Lebesgue-measurable.

The Raisonnier Filter

Corollary (Raisonnier-Shelah)

$\mathbf{\Sigma}^1_3(\mathsf{LM}) \;\Rightarrow\; \forall a \in \mathbb{R}(\aleph_1^{L[a]} < \aleph_1)$

Proof

Assume that $\mathbf{\Sigma}^1_3(\mathsf{LM})$ is true but $\exists a \in \mathbb{R}(\aleph_1^{L[a]} = \aleph_1)$. Since $2^\omega \cap L[a]$ is $\mathbf{\Sigma}^1_2$, the Raisonnier Filter $\mathcal{F}_{L[a]}$ is $\mathbf{\Sigma}^1_3$. Moreover, both other assumptions are satisfied, so $\mathcal{F}_{L[a]}$ is a rapid filter, so $\neg\mathbf{\Sigma}^1_3(\mathsf{LM})$. Contradiction. □

New Applications of the Raisonnier Filter

- I am interested in a variety of regularity properties for sets of reals in the low projective hierarchy.
- A rich structure theory exists on the $\mathbf{\Sigma}^1_2$- and $\mathbf{\Delta}^1_2$-level, reflecting the structure of cardinal invariants.
- Very little known at higher projective levels (not assuming Large Cardinals beyond an Inaccessible).
- Common theme: does the structure theory on the 2nd level lift to higher levels?

Regularity Properties on the 2nd Level

$$
\begin{array}{ccccccccc}
 & & \mathbf{\Delta}^1_2(\mathsf{LM}) & \Longrightarrow & \cdot & \Longrightarrow & \cdot & \Longrightarrow & \cdot & \Longrightarrow & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{S}) \\ \mathbf{\Delta}^1_2(\mathbb{S})\end{array} \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & & & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{L}) \\ \mathbf{\Delta}^1_2(\mathbb{L})\end{array} & \Longrightarrow & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{M}) \\ \mathbf{\Delta}^1_2(\mathbb{M})\end{array} & & & & \\
 & & & & \Uparrow & & \Uparrow & & & & \\
\forall a\,(\aleph_1^{L[a]} < \aleph_1) & \Longrightarrow & \mathbf{\Sigma}^1_2(\mathsf{LM}) & \Longrightarrow & \mathbf{\Sigma}^1_2(\mathsf{BP}) & \Longrightarrow & \mathbf{\Delta}^1_2(\mathsf{BP}) & \Longrightarrow & \cdot & &
\end{array}
$$

Here $\mathbb{L}, \mathbb{M}$ and $\mathbb{S}$ refer to Laver-, Miller- and Sacks-measurability (also called "Marczewski-Burstin algebras").

Ramsey Property and Laver-Measurability

Definition

A set $A \subseteq [\omega]^\omega$ is Ramsey if there exists a set $H \in [\omega]^\omega$ such that $[H]^\omega \subseteq A$ or $[H]^\omega \cap A = \varnothing$.

Definition

A set $A \subseteq \omega^\omega$ is Laver-measurable if there exists a Laver-tree T such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

A Laver tree $T \subseteq \omega^{<\omega}$ has a stem σ and all τ extending σ are ω-splitting.

Equivalence between $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$

Theorem (Judah-Shelah 1989)

$\boldsymbol{\Sigma}^1_2$(Ramsey) $\Leftrightarrow$ $\boldsymbol{\Delta}^1_2$(Ramsey)

Theorem (Brendle-Löwe 1999)

$\boldsymbol{\Sigma}^1_2$(Laver) $\Leftrightarrow$ $\boldsymbol{\Delta}^1_2$(Laver)

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Is this also true at higher projective levels?

Non-equivalences at higher levels

Theorem (Fischer-Friedman-K. 2014)

$\text{Con(ZFC)} \rightarrow \text{Con}(\boldsymbol{\Sigma}^1_3(\text{Ramsey}) \not\Leftrightarrow \boldsymbol{\Delta}^1_3(\text{Ramsey}))$

Theorem (Brendle-K., unpublished)

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Theorem (Fischer-Friedman-K. 2014)

$\mathsf{Con}(\mathsf{ZFC} + \mathsf{Inaccessible}) \rightarrow \mathsf{Con}(\mathbf{\Sigma}^1_4(\mathsf{Ramsey}) \not\Leftrightarrow \mathbf{\Delta}^1_4(\mathsf{Ramsey}))$

Theorem (Brendle-K., unpublished)

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Construction methods with Suslin$^+$ proper iterations

The technical side is based on our work about obtaining regularity for $\mathbf{\Delta}^1_3$ and $\mathbf{\Delta}^1_4$ sets of reals by iterating Suslin$^+$ Proper Forcing.

Sample of the techniques (Fischer-Friedman-K.)

(1) Suppose $\mathbb{AP}$ is a "quasi-amoeba" for $\mathbb{P}$, and both are Suslin$^+$ proper. Then $V^{(\mathbb{P} * \mathbb{AP})_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P}\text{-measurability})$.

(2) Suppose $V \models \forall a\, (\aleph_1^{L[a]} < \aleph_1)$ and $\mathbb{P}$ is Suslin$^+$ proper. Then $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P})$.

(3) Suppose $V \models \forall a\, (\aleph_1^{L[a]} < \aleph_1)$ and $\mathbb{AP}$ is a "quasi-amoeba" for $\mathbb{P}$, and both are Suslin$^+$ proper. Then $V^{(\mathbb{P} * \mathbb{AP})_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{P}\text{-measurability})$.

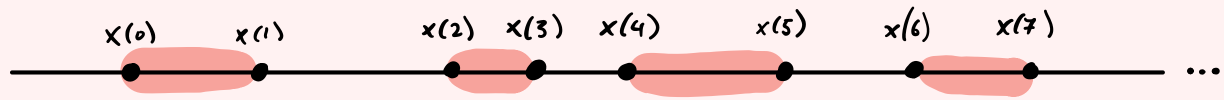
In fact, these techniques are very robust and allow combining various forcings in the iteration.

Rapid Filters and Ramsey Property

Definition

For any $x \subseteq \omega$, define

$$\overset{\approx}{x} = [x(0), x(1)) \cup [x(2), x(3)) \cup \dots$$



For a filter $\mathcal{F}$ on ω, let $\overset{\approx}{\mathcal{F}} = \{x \subseteq \omega \mid \overset{\approx}{x} \in F\}$.

Theorem (Mathias 1974)

If $\mathcal{F}$ is a rapid filter then $\overset{\approx}{\mathcal{F}}$ is not Ramsey (Mathias called it 'rare filter').

Corollary

If there is a $\mathbf{\Sigma}^1_n$ rapid filter then $\neg\mathbf{\Sigma}^1_n$(Ramsey).

Rapid Filters and Laver Property

Theorem (Brendle, unpublished)

If $\mathcal{F}$ is a rapid filter then $\widetilde{\widetilde{\widetilde{\mathcal{F}}}}$ is not Laver-measurable.

Corollary

If there is a $\mathbf{\Sigma}^1_n$ rapid filter then $\neg\mathbf{\Sigma}^1_n(\text{Laver})$.

Proof for level 3

Theorem (Fischer-Friedman-K. 2014)

$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con}(\mathbf{\Sigma}^1_3(\mathrm{Ramsey}) \not\Leftarrow \mathbf{\Delta}^1_3(\mathrm{Ramsey}))$

Theorem (Brendle-K., unpublished)

$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con}(\mathbf{\Sigma}^1_3(\mathrm{Laver}) \not\Leftarrow \mathbf{\Delta}^1_3(\mathrm{Laver}))$

Proof

(1) Start in L and iterate Mathias forcing with Amoeba-for-measure forcing.

Then we get $\Delta^1_3(\mathrm{Ramsey})$ and $\Sigma^1_2(\mathrm{LM})$, so the Raisonnier Filter $\mathcal{F}_L$ is a $\mathbf{\Sigma}^1_3$ rapid filter. So $\neg\mathbf{\Sigma}^1_3(\mathrm{Ramsey})$.

(2) Replace Mathias by Laver.

(In fact both can be done in one model).

Proof for level 4

Theorem (René David 1983)

Consistently with ZFC+Inaccessible, there exists a model L^D satisfying $\forall a(\aleph_1^{L[a]} < \aleph_1)$ and there is a $\mathbf{\Sigma}^1_3$-good wellorder of the reals.

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Consistently with ZFC+Inaccessible, there exists a model L^D satisfying $\forall a(\aleph_1^{L[a]} < \aleph_1)$ and there is a $\mathbf{\Sigma}^1_3$-good wellorder of the reals.

Observation

Suppose M is a model such that $|2^\omega \cap M| = \aleph_1$ and for all $a \in 2^\omega$, there is a measure-one set of random reals over $M[a]$. Then the Raisonnier Filter $\mathcal{F}_M$ is rapid.

Proof for level 4

Theorem (Fischer-Friedman-K. 2014)

Con(ZFC + Inaccessible) $\rightarrow$ Con($\mathbf{\Sigma}^1_4$(Ramsey) $\not\Leftarrow$ $\mathbf{\Delta}^1_4$(Ramsey))

Theorem (Brendle-K., unpublished)

Con(ZFC + Inaccessible) $\rightarrow$ Con($\mathbf{\Sigma}^1_4$(Laver) $\not\Leftarrow$ $\mathbf{\Delta}^1_4$(Laver))

Proof

(1) Start in L and iterate and iterate Mathias forcing with Amoeba-for-measure.Then we get Δ^1_4(Ramsey), and moreover the assumption above holds with respect to L^D. So the Raisonnier Filter $\mathcal{F}_{(L^D)}$ is a $\mathbf{\Sigma}^1_4$ rapid filter. So $\neg\mathbf{\Sigma}^1_4$(Ramsey).

(2) Replace Mathias by Laver.

(In fact both can be done in one model).

More things...

Theorem (K-Fischer-Friedman; Brendle)

For all standard regularity properties $\mathbb{P}$ (except Sacks and Miller),

- Consistently with ZFC, $\mathbf{\Sigma}^1_2(\mathbb{P}) + \neg\mathbf{\Sigma}^1_3(\mathbb{P})$.
- Consistently with ZFC + Inaccessible, $\mathbf{\Sigma}^1_3(\mathbb{P}) + \neg\mathbf{\Sigma}^1_4(\mathbb{P})$.

Question

However we don't know how to make models (without large cardinals) for

$$\mathbf{\Sigma}^1_4(\mathbb{P}) + \neg\mathbf{\Sigma}^1_5(\mathbb{P})$$

and in general

$$\mathbf{\Sigma}^1_n(\mathbb{P}) + \neg\mathbf{\Sigma}^1_{n+1}(\mathbb{P}).$$

More questions

Questions

- Is a rapid filter a counterexample to Sacks- and Miller-measurability?
- Is $\mathbf{\Delta}^1_3$ and $\mathbf{\Sigma}^1_3$ equivalent for Sacks- and Miller-regularity? What about $\mathbf{\Delta}^1_4$ and $\mathbf{\Sigma}^1_4$?
- Can Raisonnier's method be adapted (in any way) to replace the assumption $\mathbf{\Sigma}^1_2(\mathrm{LM})$ by another property?
- Can we get "all sets of reals are Ramsey" without assuming an inaccessible?
- Can we get "all sets of reals are Laver-measurable" without assuming an inaccessible?

Thank you!

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A. R. D. Mathias.
A remark on rare filters.
In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. III*, pages 1095–1097. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

Jean Raisonnier.
A mathematical proof of S. Shelah's theorem on the measure problem and related results.
Israel J. Math., 48(1):48–56, 1984.

Saharon Shelah.
Can you take Solovay's inaccessible away?
Israel J. Math., 48(1):1–47, 1984.

Robert M. Solovay.
A model of set-theory in which every set of reals is Lebesgue measurable.
Ann. of Math. (2), 92:1–56, 1970.