Fake nulls and meagers in the Baire space

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Measure

Measure in the Cantor space

Definition

A set $F \subseteq 2^\omega$ is null if

$$(\forall \varepsilon > 0)\,(\exists (\sigma_n)_{n\in\omega})\left(\sum_{n\in\omega} \frac{1}{2^{|\sigma_n|}} < \varepsilon \wedge F \subseteq \bigcup_{n\in\omega} [\sigma_n]\right).$$

Lemma

Let $F \subseteq 2^\omega$ be a null set. Then there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq 2^n$, $\sum \frac{|S_n|}{2^n} < \infty$, satisfying a condition

$$F \subseteq \{x \in 2^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$

Fake measure in the Baire space

Definition

A set $F \subseteq \omega^\omega$ is fake-null ($F \in f\mathcal{N}$) if

$$(\forall \varepsilon > 0)\,(\exists (\sigma_n)_{n\in\omega})\left(\sum_{n\in\omega} \frac{1}{2^{|\sigma_n|}} < \varepsilon \wedge F \subseteq \bigcup_{n\in\omega} [\sigma_n]\right).$$

Lemma

Let $F \subseteq \omega^\omega$ be a fake-null set. Then there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq \omega^n$, $\sum \frac{|S_n|}{2^n} < \infty$, satisfying a condition

$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$

Small and closed null sets

Definition

$F \in \mathcal{E}$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq 2^{I_n}$, $\sum \frac{|J_n|}{2^{|I_n|}} < \infty$

$$F \subseteq \{x \in 2^\omega : (\forall^\infty n)(x{\restriction}I_n \in J_n)\}.$$

Definition

$F \in \mathcal{S}$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq 2^{I_n}$, $\sum \frac{|J_n|}{2^{|I_n|}} < \infty$

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Fake small and closed null sets

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$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x \restriction I_n \in J_n)\}.$$

Observations

- $f\mathcal{N}$ is a translation-invariant $\sigma-$ideal
- $f\mathcal{E} \subsetneq f\mathcal{S}$
- $f\mathcal{N} \perp \mathcal{M}$
- every fake null is a union of two fake smalls

Parametrization

For now let $h \in \omega^\omega$, $\limsup_n h(n) = \infty$.

Definition

$F \in f\mathcal{N}(h)$ if there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq \omega^n$, $\sum \frac{|S_n|}{h(n)} < \infty$, satisfying a condition

$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$

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Definition

$$f\mathcal{N}(\mathrm{Fin}) = \bigcup_h f\mathcal{N}(h)$$

Parametrization

Definition

$F \in f\mathcal{E}$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$

$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \in J_n)\} .$$

Definition

$F \in f\mathcal{S}$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$

$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x \restriction I_n \in J_n)\} .$$

Observations

- $f\mathcal{N}(h)$ is a translation-invariant $\sigma-$ideal
- $h \leq g \implies f\mathcal{N}(h) \subseteq f\mathcal{N}(g)$
- $k < l \implies f\mathcal{N}(k^n) \subsetneq f\mathcal{N}(l^n)$
- $f\mathcal{E}(h) \subsetneq f\mathcal{S}(h)$
- $f\mathcal{N}(h) \perp \mathcal{M}$
- $f\mathcal{S}(h) \not\subseteq f\mathcal{N}(\mathrm{Fin})$
- $f\mathcal{N}(\mathrm{Fin})$ is not ccc
- $f\mathcal{E}(\mathrm{Fin}) = \mathcal{K}_\sigma$

Category

Fake meager

Definition

$F \in \mathcal{M}_-$ if there are $x_F \in \omega^\omega$ and a partition of ω into intervals $(I_n)_{n\in\omega}$ such that

$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \neq x_F \restriction I_n)\}.$$

Observations

- $\mathcal{M}_-$ is a translation invariant $\sigma-$ideal
- $\mathcal{M}_- \subsetneq \mathcal{M}$
- $f\mathcal{N}(\mathrm{Fin}) \not\subseteq \mathcal{M}_i$

Another meager-like ideals

Definition

For $f : \omega^{<\omega} \to \omega$ let

$$D_f = \{x \in \omega^\omega : (\forall^\infty n)(x(n) \neq f(x{\restriction}n))\}.$$

Denote $\mathcal{D}_\omega = \{A \subseteq \omega^\omega : A \subseteq D_f$ for some f$\}$.

Definition

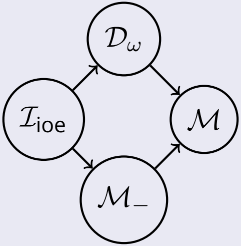
For $y \in \omega^\omega$ let

$$K_y = \{x \in \omega^\omega : (\forall^\infty n)(y(n) \neq x(n))\}.$$

Denote by $\mathcal{I}_{\text{ioe}}$ the $\sigma-$ideal generated by K_y.

Cardinal invariants

Theorem



Theorem (Khomskii, Laguzzi)

$\operatorname{cov}(\mathcal{I}_{\text{ioe}}) = \operatorname{cov}(\mathcal{D}_\omega) = \operatorname{cov}(\mathcal{M})$ *and*
$\text{non}(\mathcal{I}_{\text{ioe}}) = \text{non}(\mathcal{D}_\omega) = \text{non}(\mathcal{M})$.

Corollary

$\operatorname{cov}(\mathcal{M}_-) = \operatorname{cov}(\mathcal{M})$ *and* $\text{non}(\mathcal{M}_-) = \text{non}(\mathcal{M})$.

References

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Thank You for attention