Destruction and +-destruction of ideals

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Joint work with Barnabas Farkas and Lyubomyr Zdomskyy

Let $\mathcal{J}$ be an definable ideal on ω, preferably analytic, Borel or F_σ.

$\mathcal{J}$ is **tall** if every infinite $X \subseteq \omega$ contains infinite $Y \in \mathcal{J}$.

$\mathcal{J}$ is a **P − ideal** if for every $\{A_n : n \in \omega\} \subseteq \mathcal{J}$ there is $A \in \mathcal{J}$ such that $A_n \subseteq^* A$ for all $n \in \omega$.

$\mathrm{add}^*(\mathcal{J}) = min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \neg \exists X \in \mathcal{J} \forall F \in \mathcal{F} F \subseteq^* X\}$
$\mathrm{add}^*(\mathcal{J}) > \omega$ iff $\mathcal{J}$ is a P-ideal

Theorem

For an ideal $\mathcal{J}$ on ω we have:

- (Mazur) $\mathcal{J}$ is F_σ iff $\mathcal{J}$
  $\mathcal{J} = Fin(\phi) := \{A \subseteq \omega : \phi(A) < \infty\}$,
- (Solecki) $\mathcal{J}$ is analytic P-ideal iff
  $\mathcal{J} = Exh(\phi) := \{A \subseteq \omega : lim_n \phi(A \setminus n) = 0\}$,

for some l.s.c.s.m. $\phi : \mathcal{P}(\omega) \to [0, \infty]$.

Covering and +-covering

For tall ideal consider the following two invariants:

$$\mathrm{cov}^*(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J}\ \forall X \in [\omega]^\omega\ \exists F \in \mathcal{F}\ |X \cap F| = \omega\},$$

$$\mathrm{cov}^*_+(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J}\ \forall X \in \mathcal{J}^+\ \exists F \in \mathcal{F}\ |X \cap F| = \omega\}$$

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$\mathrm{add}^*(\mathcal{J}) \le \mathrm{cov}^*_+(\mathcal{J}) \le \mathrm{cov}^*(\mathcal{J})$

For example, for the ideal $nwd = \{A \subseteq \mathbb{Q} : \bar{A}$ is nowhere dense$\}$ we have:

$$\mathrm{add}^*(\mathcal{J}) = \omega,$$

$$\mathrm{cov}^*_+(\mathcal{J}) = \mathrm{add}(\mathcal{M}),$$

$$\mathrm{cov}^*(\mathcal{J}) = \mathrm{cov}(\mathcal{M}).$$

Katetov reduction

If $\mathcal{J}$ and $\mathcal{I}$ are ideals on ω then we write $\mathcal{J} \leq_K \mathcal{I}$ if there is $\pi : \omega \to \omega$ such that $\pi^{-1}[X] \in \mathcal{I}$ for every $X \in \mathcal{J}$.

If $\mathcal{J} \leq_K \mathcal{I}$ then

- $\mathrm{cov}^*(\mathcal{I}) \leq \mathrm{cov}^*(\mathcal{J})$,
- $\mathrm{cov}^*_+(\mathcal{I}) \leq \mathrm{cov}^*_+(\mathcal{J})$.

Destruction and +destruction

Suppose that $\mathbb{P}$ is a notion of forcing and $\mathcal{J}$ is an ideal on ω.
We say that:

- $\mathbb{P}$ destroys $\mathcal{J}$ if it adds infinite $x \subseteq \omega$ such that $|x \cap F| < \omega$ for every ground model $F \in \mathcal{F}$,
- $\mathbb{P}$ +-destroys $\mathcal{J}$ if it adds $\mathcal{J}$-positive $x \subseteq \omega$ such that $|x \cap F| < \omega$ for every ground model $F \in \mathcal{F}$

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Typical forcing notions for destruction of $\mathcal{J}$'s:

- Mathias-Prikry with dual filter $\mathbb{M}(\mathcal{J}^*)$,
- Laver-Prikry with dual filter $\mathbb{L}(\mathcal{J}^*)$,
- Laver or Mathias with $\mathcal{J}$-positive sets,
- Hrušák fat forcing (for F_σ-ideals),

Theorem (Farkas, Zdomskyy)

The following are equivalent for tall Borel ideal $\mathcal{J}$:

– $\mathbb{M}(\mathcal{J}^*)$ +-destroys $\mathcal{J}$,

– there is $\mathbb{P}$ which +-destroys $\mathcal{J}$,

– $\mathrm{cov}^*_+(\mathcal{J})$ is uncountable.

Theorem (Farkas, Zdomskyy)

The following are equivalent for analytic P ideal $\mathcal{J}$:

– $\mathbb{L}(\mathcal{J}^*)$ +-destroys $\mathcal{J}$,

– $\mathcal{J}$ is fragile.

More examples

F_σ-examples

- fin,
- random graph ideal $\mathcal{R}$,
- Solecki ideal $\mathcal{S}$,
- summable ideal $\mathcal{I}_{\frac{1}{n}}$,
- eventually different ideal $\mathcal{ED}_{fin}$.

$\mathcal{ED}_{fin}$-ideal

Definition (Hernández-Hernández and Hrušák)

Let $\mathcal{ED}_{fin}$ be the ideal on the countable set

$$\Delta = \{(i,j) \in \omega \times \omega : j \leq i\}$$

made of these $X \subseteq \Delta$ s.t. for some $N \in \omega$, almost all vertical sections of X are of size $\leq N$.

- $\mathrm{cov}^*_+(\mathcal{ED}_{fin}) = \mathrm{cov}^*(\mathcal{ED}_{fin})$
- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}^*(\mathcal{ED}_{fin}) \leq \mathrm{non}(\mathcal{M})$
- $\mathrm{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\}$,
- $\mathcal{ED}_{fin} \leq_K \mathcal{J}$ for every tall analytic P-ideal

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analytic P-ideals:

- summable ideal $\mathcal{I}_{\frac{1}{n}}$,
- trace of measure ideal $tr(\mathcal{N})$,
- density zero ideal $\mathcal{Z}$.

Examples

$tr(\mathcal{N})$ ideal

For $A \subseteq 2^{<\omega}$ let

$$\pi[A] = \{x \in 2^\omega : \exists_n^\infty x|_n \in A\}$$

be the G_δ-closure of A. Let then $tr(\mathcal{N}) = \{A \subseteq 2^{<\omega} : \pi[A] \in \mathcal{N}\}$.

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- $\mathcal{ED}_{fin} \leq_K \mathcal{I}_{\frac{1}{n}} \leq_K tr(\mathcal{N})$,
- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}^*(tr(\mathcal{N})) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ (Hrušák, Zapletal)

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- $\mathcal{J} \leq_K tr(\mathcal{N})$ iff random forcing destroys $\mathcal{J}$ (Brendle, Yatabe)

Examples

Density zero ideal

Let $\mathcal{Z}$ be the ideal on these $A \subseteq \omega$ such that $lim_n \frac{|A \cap n|}{n} = 0$.
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- $\mathrm{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$ (Hernández-Hernández, Hrušák),

- $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}^*(\mathcal{Z})$ (Hernández-Hernández, Hrušák).

Summarizing, we have

$$\mathcal{ED}_{fin} \leq_K \mathcal{I}_{\frac{1}{n}} \leq_K tr(\mathcal{N}) \leq_K \mathcal{Z}$$

and thus

$$\mathrm{cov}(\mathcal{Z}) \leq \mathrm{cov}(tr(\mathcal{N})) \leq \mathrm{cov}(\mathcal{I}_{\frac{1}{n}}) \leq \mathrm{cov}(\mathcal{ED}_{fin}) \leq \mathrm{non}(\mathcal{M})$$

Connections to closed measure zero

$\mathcal{E}$ is the σ-ideal generated by closed, measure zero subsets of 2^ω.

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It is known that

- $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$,
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Theorem(C.-Farkas-Zdomskyy)

$\mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) \leq \mathrm{non}(\mathcal{E}) \leq \mathrm{non}^*_+(tr(\mathcal{N}))$,
$\mathrm{cov}^*_+(tr(\mathcal{N})) \leq \mathrm{cov}(\mathcal{E}) \leq \mathrm{non}^*_+(\mathcal{I}_{\frac{1}{n}})$.

Recall well-known characterizations of measure and category from Bartoszyński and Miller:

Theorem (Bartoszyński)

$\mathrm{add}(\mathcal{N})$ is equal to:
$\min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \neg\exists S \in \mathcal{S}lm \forall f \in \mathcal{F}\ f \subseteq^* S\}$

Theorem (Bartoszyński-Miller)

$\mathrm{non}(\mathcal{M})$ is equal to:
$\min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega\ \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ f =^\infty g\}$ and
$\min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{S}lm\ \forall g \in \omega^\omega\ \exists S \in \mathcal{F}\ f \in^\infty S\}$

Where $S \in \mathcal{S}$ if $S : \omega \to [\omega]^{<\omega}$ is s.t. $|S(n)| = n$.

Relation systems

A triple $\mathcal{R} = (\mathcal{X}, \sqsubseteq, \mathcal{Y})$ is a relation system if

- $\mathcal{X}$,$\mathcal{Y}$ are non-empty sets,
- $\sqsubseteq$ is relation between $\mathcal{X}$ and $\mathcal{Y}$

Define then:

- The (un)bounding number
  $\mathfrak{b}(\mathcal{R}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{X} \ \neg\exists y \in \mathcal{Y} \forall x \in \mathcal{F} \ x \leq y\}$
- The dominating number
  $\mathfrak{d}(\mathcal{R}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{Y} \ \forall x \in \mathcal{X} \exists y \in \mathcal{F} \ x \leq y\}$

In this way:
$\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\omega^\omega, \subseteq^*, Slm)$
$\mathrm{non}(\mathcal{M}) = \mathfrak{d}(\omega^\omega, =^\infty, \omega^\omega) = \mathfrak{d}(\omega^\omega, \in^\infty, \mathcal{S}lm)$

What happens if we take bounded versions of the above?

We get:
$\mathfrak{b}(\prod F, \subseteq^*, SIm(F, b))$
$\mathfrak{d}(\prod F, =^\infty, \prod F)$
$\mathfrak{d}(\prod F, \in^\infty, \mathcal{S}lm(F, b))$

Where

- $\prod F := \prod_n F(n)$ for rather fast $F \in \omega^\omega$
- $S \in \mathcal{S}lm(F, b)$ if $|S(n)| \leq b(n)$ and $\sum_n \frac{b(n)}{F(n)} < \infty$

We have that:

$\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\Pi F, \subseteq^*, Slm(F,b)) \leq \mathfrak{d}(\Pi F, =^\infty, \Pi F) \leq \mathrm{non}(\mathcal{M})$,

$\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}(\Pi F, \in^\infty, \mathcal{S}lm(F,b)) \leq \mathfrak{d}(\Pi F, =^\infty, \Pi F) \leq \mathrm{non}(\mathcal{M})$

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Actually

- $\mathfrak{d}(\prod F, =^\infty, \prod F) \leq \mathrm{cov}^*(\mathcal{ED}_{fin})$,
- $\mathfrak{d}(\prod F, \in^\infty, \mathcal{S}lm(F,b)) \leq \mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}})$,
- $\mathfrak{b}(\prod F, \subseteq^*, Slm(F,b)) \leq \mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}})$

Thank you