Universal Borel graphs under homomorphism

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Joint work with Zoltán Vidnyánszky.

Definitions

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Example

The irrational rotation graph: $V(G) = S^1$, and let $\alpha \in [0, \pi)$ be irrational. Denote by T_α the rotation of the circle, and let $(x, y) \in E(G) \iff T_\alpha(x) = y$ or $T_\alpha(y) = x$.

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Definition (Borel chromatic number)

The Borel chromatic number of a graph G, denoted by $\chi_B(G)$ is the minimal $n \in \{1, 2, ..., \aleph_0\}$, such that G admits a Borel n-coloring, that is a Borel map $c : V(G) \to n$ with $\forall x, y \in V(G) : (x, y) \in E(G) \implies c(x) \neq c(y)$.

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Let G, H be two graphs. We call a function $\varphi : V(G) \to V(H)$ a homomorphism, if $\forall x, y \in V(G) : (x, y) \in E(G) \implies (\varphi(x), \varphi(y)) \in E(H)$.

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Definition (Hyperfiniteness)

A Borel graph G is hyperfinite, if the connectedness equivalence relation of G is hyperfinite.
(A countable Borel equivalence relation (CBER) E is hyperfinite, there are CBERs $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ with finite classes such that $E = \bigcup_{n \in \mathbb{N}} E_n$.)

The graph G_0

Definition (Graph G_0)

Let $s_n \in 2^n$ be chosen for every $n \in \mathbb{N}$ such that $\forall s \in 2^{<\mathbb{N}}\ \exists n\ s \sqsubseteq s_n$. Then define the graph G_0 on $2^{\mathbb{N}}$ as:

$$G_0 = \{(s_n{}^\frown 0^\frown x, s_n{}^\frown 1^\frown x), (s_n{}^\frown 1^\frown x, s_n{}^\frown 0^\frown x) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : n \in \mathbb{N}, x \in 2^{\mathbb{N}}\}.$$

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Fact

$\chi_B(G_0) > \aleph_0$.

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Theorem (G_0 dichotomy, Kechris-Solecki-Todorčević [1])

Suppose G is a Borel graph on a standard Borel space X. Then exactly one of the following holds:

- there is a Borel homomorphism from G_0 to G,

- $\chi_B(G) \leq \aleph_0$.

The shift-graph

Definition (Shift-graph)

The shift-graph, G_S on $[\mathbb{N}]^{\mathbb{N}}$ is defined as the symmetrization of the graph of the shift-map S, that is, $S(x) = x \setminus \{\min x\}$.

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Fact

$\chi_B(G_S) = \aleph_0$.

Theorem (Kechris-Solecki-Todorčević [1])

Let $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ be Borel. Then $\chi_B(G_S \restriction C) \in \{1, 2, 3, \infty\}$.

Do we have a G_S dichotomy?

- Would be nice to have:

FALSE

Suppose G is a Borel graph on a standard Borel space X. Then exactly one of the following holds:
- there is a Borel homomorphism from G_S to G,
- $\chi_B(G) < \aleph_0$.

Do we have a G_S dichotomy?

- Maybe:

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Let $B \subseteq [\mathbb{N}]^{\mathbb{N}}$ be a Borel subset. Then exactly one of the following holds:
- there is a Borel homomorphism from G_S to $G_S \upharpoonright B$,
- $\chi_B(G_S \upharpoonright B) < \aleph_0$.

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False due to complexity results of Todorčević and Vidnyánszky [3].

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Conjecture

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- there is a Borel homomorphism from G_S to $G_S \restriction B$,
- $\chi_?(G_S \restriction B) < \aleph_0$.

New theorems

Theorem (K-Vidnyánszky [2], 2025+)

Let $f : X \to X$ be an acyclic Borel function on the standard Borel space X. Then there is a Borel homomorphism from the assosiated graph, G_f to the shift graph G_S.

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Let $f : X \to X$ be an acyclic Borel function on the standard Borel space X. Then there is a Borel homomorphism from the assosiated graph, G_f to the shift graph G_S.

Theorem (K-Vidnyánszky [2], 2025+)

Let G be an acyclic, hyperfinite Borel graph on a standard Borel space. Then there is a Borel homomorphism from G to G_0.

Proof sketch I.

Theorem (K-Vidnyánszky [2], 2025+)

Let $f : X \to X$ be an acyclic Borel function on the standard Borel space X. Then there is a Borel homomorphism from the assosiated graph, G_f to the shift graph G_S.

Proof.

- Reducing to the case when f is the shift map on $2^{\mathbb{N}}$,

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Proof.

- Reducing to the case when f is the shift map on $2^{\mathbb{N}}$,
- proving the statement for the shift map on $2^{\mathbb{N}}$, with toast.

□

Proof sketch I.

Reduction part.

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Reduction part.

- let $c : X \to \mathbb{N}$ be an $\mathbb{N}$-coloring,
- components with strictly monotone increasing forward orbit ☺
- on the rest let us define a function $a : X \to \{0, 1\}$ as:

$$a(x) = \begin{cases} 1 \text{ if } c(x) < c(f(x)) \\ 0 \text{ if } c(x) > c(f(x)). \end{cases} \qquad ☺$$

□

Finishing proof I.

Definition (Toast)

Given a Borel graph G, we say that a Borel collection $\mathcal{T}$ of finite subsets of V(G) is an r-toast if it satisfies

1. $\bigcup_{K\in\mathcal{T}} E(K) = E(G)$

Finishing proof I.

Definition (Toast)

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1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$
2. for every pair $K, L \in \mathcal{T}$ either $N_r(K) \cap N_r(L) = \emptyset$ or $N_r(K) \subseteq L$ or $N_r(L) \subseteq K$,

where $N_r(X)$ is the r-neighbourhood of X using the graph distance.

Proof sketch II.

Theorem (K-Vidnyánszky [2], 2025+)

Let G be an acyclic, hyperfinite Borel graph on a standard Borel space. Then there is a Borel homomorphism from G to G_0.

Proof.

- Embed homomorphically every Borel graph into an inverse limit,
- then prove the theorem for inverse limits.

□

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Thank you for your attention!