Subgroups of Big Mapping Class Groups That Are Not Extremely Amenable

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Definition (Extreme Amenability)

A topological group is **extremely amenable** if in any Hausdorff compact space on which it continuously acts admits a global fixed point.

Theorem (Cuauhti-Hernández-Morales-Navarro-Ramos-Randecker)

*A subgroup Γ of a big mapping class group $\mathrm{Map}^{\pm}(\Sigma)$ is **not** extremely amenable if*

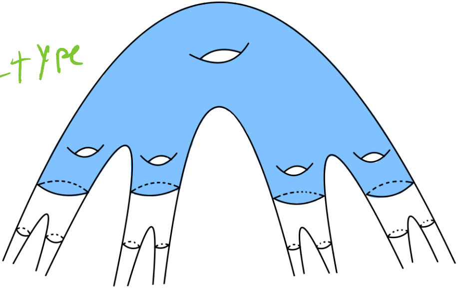
1. *there exist mapping classes $g_1, \ldots, g_n \in \overline{\Gamma}$ and a **two-sided curve** α such that the set $\{\alpha, g_1(\alpha), \ldots, g_n(\alpha)\}$ **bounds** a finite-type subsurface of Σ, and*
2. *for any $f \in \Gamma$ the subsurfaces bounded by the sets $\{f(\alpha), fg_1(\alpha), \ldots, fg_n(\alpha)\}$ and $\{\alpha, g_1(\alpha), \ldots, g_n(\alpha)\}$ have **homotopically disjoint interior**.*

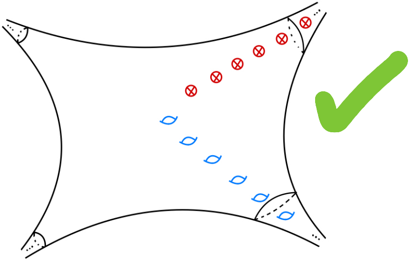
Surfaces

Definition

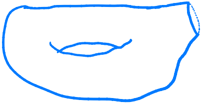
A **surface** is a connected metrizable space locally homeomorphic to $\mathbb{R}^2$.

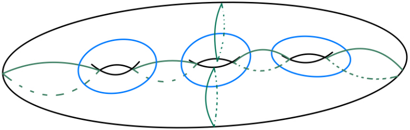
Inf-type

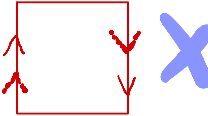




Fin-type







Definition

The **mapping class group** of a surface Σ is the topological group

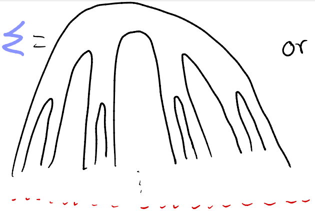
$$\mathrm{Map}(\Sigma) := \frac{\mathrm{Homeo}(\Sigma)}{\mathrm{Homeo}_o(\Sigma)},$$

where $\mathrm{Homeo}(\Sigma)$ is equipped with the **compact-open topology** and $\mathrm{Homeo}_0(\Sigma)$ is the path-connected component of Id.

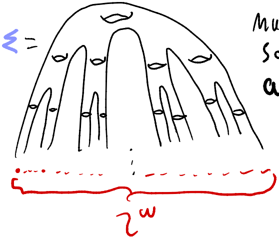
When Σ is of **infinite-type** (i.e. its π_1 is not finitely generated), $\mathrm{Map}(\Sigma)$ is known as the (extended) big mapping class group associated to Σ.

Examples of BMCGS that are not Extremely Amenable

If $\Sigma =$



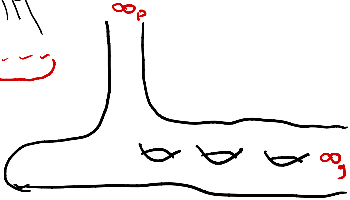
or $\Sigma =$



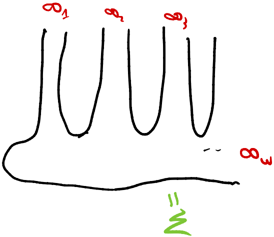
$\mathrm{Map}(\Sigma) \curvearrowright 2^\omega$ transitively, so $\mathrm{Map}(\Sigma)$ is not extremely amenable.

$\Sigma =$

$\Sigma =$



The action $\mathrm{Map}(\Sigma) \curvearrowright \mathrm{Ends}(\Sigma)$ has a fixed point, so we can't use it to prove that $\mathrm{Map}(\Sigma)$ is not extremely amenable.



Yusen Long´s Results

- In general, big mapping class groups are not extremely amenable.
- The subgroup of *compactly supported* mapping classes of infinite-type surfaces with **positive genus** are also not extremely amenable.

Our goal is to extend the list of subgroups of big mapping class groups that are not extremely amenable.

In order to do so, first we will associate to any $\Gamma \leqslant \mathrm{Map}(\Sigma)$ an **ultrahomogeneous countable structure** $C_\Gamma(\Sigma)$ with

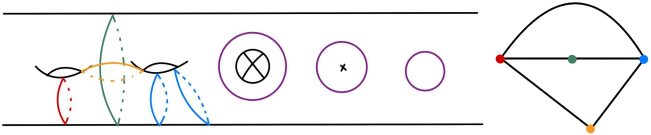
$$\mathrm{Aut}(C_\Gamma(\Sigma)) \cong \overline{\Gamma}$$

and then we will prove that several classes of finite substructures $\mathrm{Age}(C_\Gamma(\Sigma))$ do not have the Ramsey property.

Curve Graph

Definition

The **curve graph** of a surface Σ is the graph $C(\Sigma)$ whose vertices are **isotopy classes of essential simple closed curves** on Σ and adjacency is given by disjointness.

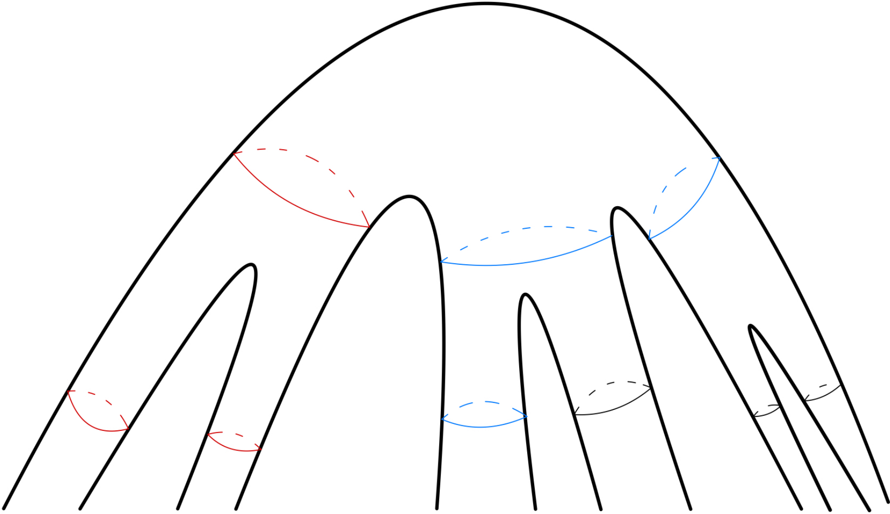


The natural map

$$\begin{aligned} \mathrm{Map}(\Sigma) &\longrightarrow \mathrm{Aut}\big(C(\Sigma)\big) \\ [f] &\longmapsto ([\alpha] \mapsto [f(\alpha)]) \end{aligned}$$

is an isomorphism of topological groups when $\mathrm{Aut}\big(C(\Sigma)\big)$ is equipped with the topology inherited by $S_\infty = \mathrm{Bij}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$.

The Curve Graph IS NOT Ultrahomogeneous

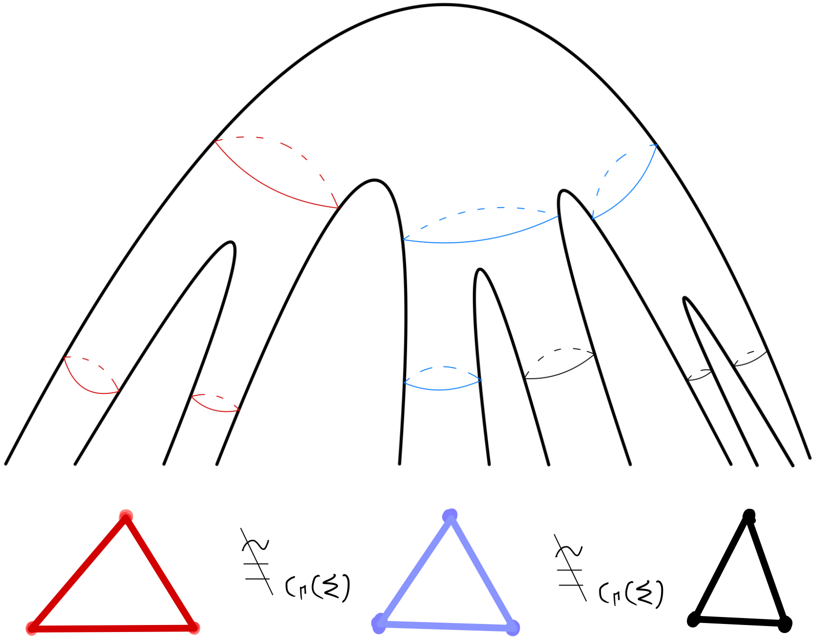


Definition

The **full curve graph** of Σ with respect to a subgroup $\Gamma \leqslant \mathrm{Map}(\Sigma)$ is the structure $C_\Gamma(\Sigma)$ obtained from $C(\Sigma)$ by adjoining for every $n \in \mathbb{N}$ the orbits of the canonical action

$$\Gamma \times C(\Sigma)^n \longrightarrow C(\Sigma)^n$$
$$([f], [\alpha_1], \cdots, [\alpha_n]) \longmapsto ([f(\alpha_1)], \cdots, [f(\alpha_n)])$$

as n-ary relations.



$$\{\alpha_1, \cdots, \alpha_n\} \cong_{C_\Gamma(\Sigma)} \{\beta_1, \cdots, \beta_n\} \text{ iff } \exists f \in \Gamma \text{ s.t } \forall i = 1, \ldots, n$$

$$f(\alpha_i) = \beta_i.$$

Theorem

For any subgroup $\Gamma \leqslant \mathrm{Map}(\Sigma)$ *the natural map*

$$\begin{aligned} \overline{\Gamma} &\longrightarrow \mathrm{Aut}\big(C_\Gamma(\Sigma)\big) \\ [f] &\longmapsto ([\alpha] \mapsto [f(\alpha)]) \end{aligned}$$

is an isomorphism of topological groups.

This theorem implies that

$$\mathrm{Age}(C_\Gamma(\Sigma)) = \{\text{finite substructures embeddable in } C_\Gamma(\Sigma)\}$$

is a Fraïssé class.

KPT Correspondence

Theorem (Kechris, Pestov and Todorčević)

The automorphism group of an ***ultrahomogeneous countable structure*** X *is extremely amenable if and only if*

1. *for any* $A \in \mathrm{Age}(X)$, $\mathrm{Aut}(A) = \{\mathrm{Id}_A\}$, *and*
2. $\mathrm{Age}(X)$ *has the* ***Structural Ramsey property****.*

Given $A \leqslant B \in \mathrm{Age}(X)$

$$\binom{B}{A} := \{A' \subseteq B \ : \ A' \cong A\}.$$

$\mathrm{Age}(X)$ satisfies the **(Structural) Ramsey property** if for every $A \leqslant B \in \mathrm{Age}(X)$ and $k \in \mathbb{N}_{\geqslant 2}, \exists C \in \mathrm{Age}(X)$ with $B \leqslant C$ s.t $\forall c \colon \binom{C}{A} \to k \ \exists B' \in \binom{C}{B}$ for which c is constant on $\binom{B'}{A}$.

Under the scope of the KPT Correspondence, Yusen Long's results indicate that there are finite substructures of the full curve graph admitting non-trivial automorphisms.

For full big mapping class groups, such finite substructures can be chosen to have only two elements.

Main Theorem of the Talk

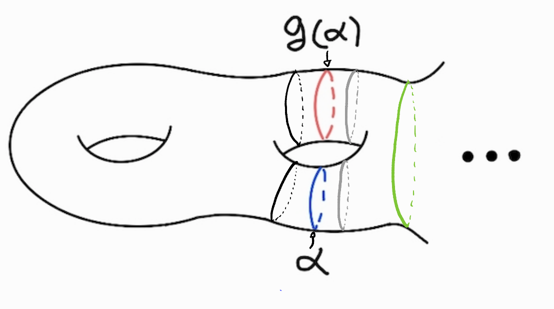
Theorem (Cuauhti-Hernández-Morales-Navarro-Ramos-Randecker)

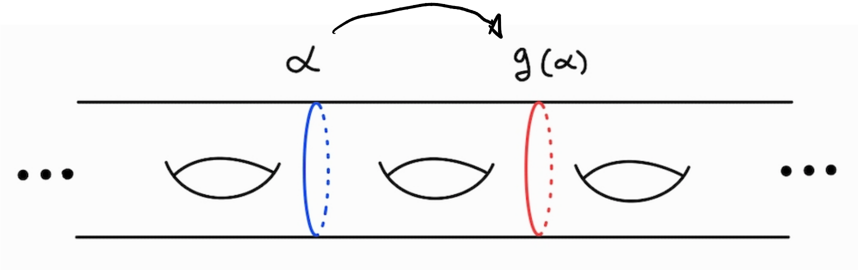
*A subgroup Γ of a big mapping class group $\mathrm{Map}^{\pm}(\Sigma)$ is **not** extremely amenable if*

1. *there exists a $g \in \overline{\Gamma}$ and a **two-sided** curve α on Σ such that the set $\{\alpha, g(\alpha)\}$ **bounds** a finite-type subsurface of Σ, and*
2. *for any $h \in \Gamma$ the subsurfaces bounded by $\{\alpha, g(\alpha)\}$ and $\{h(\alpha), h(g(\alpha))\}$ either have homotopically disjoint interior or, up to homotopy, are the same.*

We will only focus on item 1 (In a work in progress we are removing item 2).

Examples of Subgroups Fulfilling Item 1

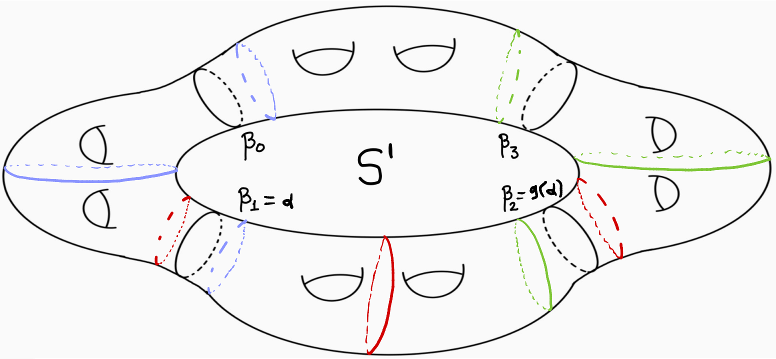




Sketch of the Proof

Lemma

Under Item 1 and Item 2 of the Main Theorem, there are no cycles in the curve graph $C(\Sigma)$ with all its edges in the Γ-orbit of $\{\alpha, g(\alpha)\}$.

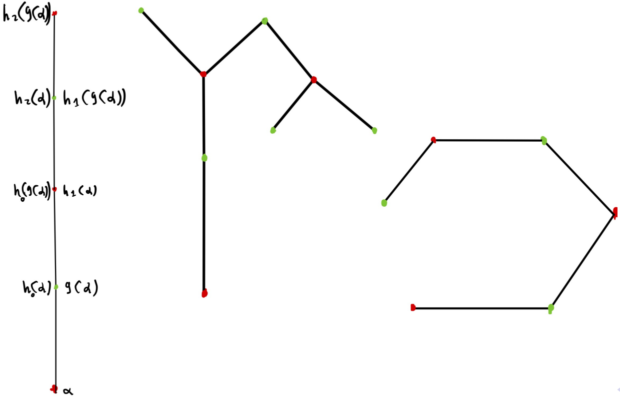


$\mathrm{Age}(C_\Gamma(\Sigma))$ does not have the Ramsey Property

Consider the parameters

- $A = \{\alpha\}$,
- $B = \{\alpha, g(\alpha)\}$,
- $k = 2$.

If $B \leqslant C \in \mathrm{Age}(C_\Gamma(\Sigma))$, we color the vertices in the union of the Γ-copies of B in C in an alternating fashion.



How could we remove Item 2?

Considering the parameters

- $A = \{\alpha\}$,
- $B = \{\alpha, g(\alpha)\}$,
- $k = 2$,

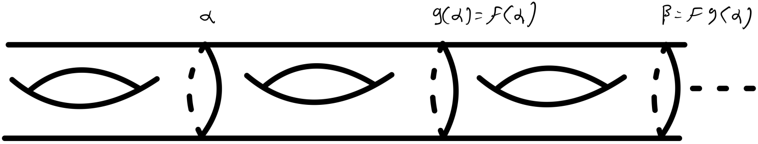
we only need conditions on $\Gamma \leqslant \mathrm{Map}(\Sigma)$ ensuring that any finite subgraph of $C(\Sigma)$ whose edges are in the Γ-orbit of $\{\alpha, g(\alpha)\}$ admits a vertex-coloring in two colors with no monochromatic edges.

Main Theorem conditions imply that any such finite subgraph is a *forest*. But to color vertices with no monocromatic edges, we only need to get rid of cycles of odd length.

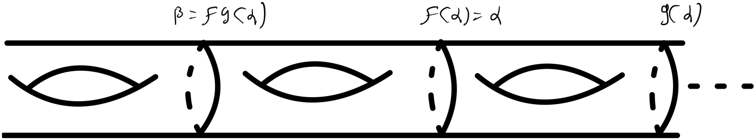
Theorem

Under Item 1 of the Main Theorem, there are no triangles in the curve graph $C(\Sigma)$ with all their edges in the Γ-orbit of $\{\alpha, g(\alpha)\}$.

Case 1 $\exists f \in \Gamma$ s.t $f(\alpha) = g(\alpha)$ and $fg(\alpha) = \beta$ but $f(\beta) \neq \alpha$



Case 2 $\exists f \in \Gamma$ s.t $f(\alpha) = \alpha$ and $fg(\alpha) = \beta$



Case 3 $\exists f \in \Gamma$ s.t $f(\alpha) = g(\alpha)$ and $fg(\alpha) = \beta$ with $f(\beta) = \alpha$