*operation and microscopic sets

Daria Perkowska

Wrocław University of Science and Technology

Let $(X, +)$ be an abelian group. For $A, B \subseteq X$ we write

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition of *operation

For a family $\mathcal{F} \subseteq \mathcal{P}(X)$ let:

$$\mathcal{F}^* = \{A \subseteq X : \forall_{F \in \mathcal{F}} A + F \neq X\}.$$

Theorem

- $\mathcal{G} \subseteq \mathcal{F}^* \Rightarrow \mathcal{F} \subseteq \mathcal{G}^*$
- $\mathcal{F} \subseteq \mathcal{F}^{**}$
- $\mathcal{G} \subseteq \mathcal{F} \Longrightarrow \mathcal{F}^* \subseteq \mathcal{G}^*$
- $\mathcal{F}^*$ is closed under taking subsets and translation invariant

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- $\forall_{A \notin \mathcal{F}} (\mathcal{F} \cup \{A\})^* \neq \mathcal{F}^*$
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Theorem

If $I = J^*$ for some family J then $I = I^{**}$

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If $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation and reflection invariant proper σ-ideal with $cof(\mathcal{J}) \leqslant \mathfrak{c}$, $add(\mathcal{J}) = \mathfrak{c}$ and $A \notin \mathcal{J}$, then

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Proof

Take family $\mathcal{F} \subseteq \mathcal{J}$ of subsets of X with $card\mathcal{F} = \mathfrak{c}$ such that for every set $I \in \mathcal{J}$ there exists a set $F \in \mathcal{F}$ covering I ($I \subseteq F$).
Let $\{z_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of 2^ω and let $\{F_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of all sets from $\mathcal{F}$. We build sequences of $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ and $\{r_\alpha\}_{\alpha<\mathfrak{c}}$.Take two different x_0 and r_0. Let $\lambda < \mathfrak{c}$. Suppose that we already constructed $\{x_\alpha\}_{\alpha<\lambda}$ and $\{r_\alpha\}_{\alpha<\lambda}$ and define x_λ and r_λ. Since $\bigcup_{\alpha_1,\alpha_2<\lambda}(F_{\alpha_1} + x_{\alpha_2}) \neq 2^\omega$, we can choose $r_\lambda \notin \bigcup_{\alpha_1,\alpha_2<\lambda}(F_{\alpha_1} + x_{\alpha_2})$.

Proof.

Let $B_\lambda = 2^\omega \setminus \bigcup_{\alpha_1,\alpha_2 \leqslant \lambda}(r_{\alpha_1} - F_{\alpha_2})$. Then $2^\omega \setminus B_\lambda \in \mathcal{J}$. Obviously $2^\omega \setminus (z_\lambda - B_\lambda) = z_\lambda - (2^\omega \setminus B_\lambda) \in \mathcal{J}$. Since $A \notin \mathcal{J}$, then $A \nsubseteq 2^\omega \setminus (z_\lambda - B_\lambda)$, so $A \cap (z_\lambda - B_\lambda) \neq \emptyset$. Hence, there are $a_\lambda \in A$ and $b_\lambda \in B_\lambda$ such that $z_\lambda - b_\lambda = a_\lambda$. Let $x_\lambda = b_\lambda$. Using this procedure, for every $\lambda < \mathfrak{c}$, we define $X = \{x_\alpha\}_{\alpha<\mathfrak{c}}$. Since $A + X = 2^\omega$, the set X does not belong to $(\mathcal{J} \cup \{A\})^*$. On the other hand, for $\alpha < \mathfrak{c}$ choosing $\lambda > \alpha$ we have $r_\lambda \notin X + F_\alpha$, so $X \in \mathcal{J}^*$.

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Question

Can we reverse this implication?

Strong measure zero sets

A set $A \subseteq \mathbb{R}$ has a strong zero measure when for every sequence (ε_n) of positive reals there exists a sequence (I_n) of intervals such that $|I_n| \leqslant \varepsilon_n$ and A is contained in the union of I_n.

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Strong measure zero sets but on Cantor

A set $A \subseteq 2^\omega$ has a strong zero measure when for every sequence (k_n) of natural numbers there exists a sequence (σ_n) of intervals such that $|\sigma_n| = kn$ and $A \subseteq \bigcup [\sigma_n]$.

Strongly meager sets

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Theorem (Galvin-Mycielski-Solovay)

A set $X \subseteq \mathbb{R}$ is strongly measure zero if and only if for every meager set H it holds that $X + H \neq \mathbb{R}$.

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$$\mathcal{SMZ} = \mathcal{M}^*$$

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Corollary

- Borel Conjecture $\Longrightarrow \mathcal{M} \neq \mathcal{M}^{**}$
- dual Borel Conjecture $\Longrightarrow \mathcal{N} \neq \mathcal{N}^{**}$

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Definition

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Definition

We say that a set $X \subseteq 2^\omega$ is microscopic ($X \in Micro$) if for all $k \in \mathbb{N}$ there exists a sequence (σ_n) such that $|\sigma_n| = kn$ and $X \subseteq \bigcup [\sigma_n]$.

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Of course, we have $\mathcal{SMZ} \subseteq Micro \subseteq \mathcal{N}$

Definition

We say that $X \subseteq \mathbb{R}$ is porous ($X \in \mathcal{P}$) if there exists $\alpha \in (0, 1)$ for every $x \in \mathbb{R}$ there exists y such that $B(y, \alpha r) \subseteq B(x, r)$ and $B(y, \alpha r) \cap X = \emptyset$

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Of course we have $\mathcal{P} \subseteq \mathcal{M}$

Theorem

$X \in Micro \implies \forall E \in \mathcal{P}\ X + E \neq 2^{\omega}$

Proof.

Theorem

If $E \in \mathcal{P}$*, then there exists* $k \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *and* $\alpha \in 2^m$ *and every* $\tau \in 2^{m+k}$*, there exists* $\beta \in 2^{m+k}$ *such that such that* $\alpha \subseteq \beta$ *and* $([\tau] + [\beta]) \cap E = \emptyset$

Theorem

$X \in Micro \Longrightarrow \forall E \in \mathcal{P}\ X + E \neq 2^\omega$

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Theorem

If $E \in \mathcal{P}$, then there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $\alpha \in 2^m$ and every $\tau \in 2^{m+k}$, there exists $\beta \in 2^{m+k}$ such that such that $\alpha \subseteq \beta$ and $([\tau] + [\beta]) \cap E = \emptyset$

Let $E \in \mathcal{P}$. Take $k \in \mathbb{N}$ from the Theorem above. Take $X \in Mic$ and the sequence of (σ_n) such that $|\sigma_n| = (n+1)k$ and $X \subseteq \bigcup[\sigma_n]$. From Theorem there exist $[\tau_n] \in 2^{(n+1)k}$ for every $[\sigma_n]$ such that $[\sigma_n] + [\tau_n] \cap E = \emptyset$. Let $y = \bigcup_n \tau_n$, then $(X + y) \cap E = \emptyset$ □