Infinite-Exponent Partition Relations on Linear Orders

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Winter School, Hejnice
31st of January, 2025

Contains joint work with Thilo Weinert and Jonathan Schilhan

Definition: Order Type

A *(linear) order type* is an isomorphism class of linear orders. We will use the letters σ and τ for arbitrary linear order types.

Addition and multiplication of order types is done exactly as for ordinals. The relation $\sigma \leq \tau$ means σ embeds into τ. The reverse of τ is denoted τ^*.

Notation: Copies of τ in $\langle L, < \rangle$

For $\langle L, < \rangle$ a linear order, τ a linear order type, write

$$[\langle L, < \rangle]^\tau := \{A \subseteq L : \langle A, < \rangle \text{ is ordered as } \tau\}.$$

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The partition relation symbol

Definition: Partition Relation Symbol

For $\langle L, <\rangle$ a linear order, σ, τ linear order types, and χ a set,

$$\langle L, <\rangle \to (\sigma)^{\tau}_{\chi}$$

is the statement that for any $F : [\langle L, <\rangle]^{\tau} \to \chi$, thought of as a *colouring* of the copies of τ in $\langle L, <\rangle$ with χ colours, there is some $H \in [\langle L, <\rangle]^{\sigma}$ which is *homogeneous* or *monochromatic* for F, in the sense that $|F \text{ " } [\langle H, <\rangle]^{\tau}| = 1$.

When χ is omitted, it is understood to equal 2.
Roughly speaking, in ZFC, relations whose exponent τ is infinite are all either false or trivial, so our base theory throughout will be ZF.

A lot of work has been done on infinite-exponent partition relations on ordinals:

Theorem (Mathias)

In Solovay's model,

$$\omega \to (\omega)^{\omega}.$$

Theorem (Martin)

In ZF + AD,

$$\omega_1 \to (\omega_1)^{\omega_1}.$$

Results on $\mathbb{R}$

Theorem (G.)

The only statements of the form $\langle \mathbb{R}, < \rangle \to (\tau)^\tau$ with τ a countably infinite order type which can hold in ZF are those in which τ is one of the order types $\omega + k$ or $k + \omega^*$ for some $k \in \omega$, and for these τ,

$$\langle \mathbb{R}, < \rangle \to (\tau)^\tau \iff \omega \to (\omega)^\omega.$$

Theorem (G.)

Relative to an inaccessible cardinal, it is consistent with ZF that $\langle \mathbb{R}, < \rangle \not\to (\tau)^\tau$ for all uncountable order types τ. Moreover, this is the case iff there are no infinite exact sets of reals.

A linear order $\langle L, < \rangle$ is *exact* if it has no proper suborder which is order-isomorphic to it.

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Larger homogeneous sets

If $\omega \to (\omega)^\omega$, we in fact have $\langle \mathbb{R}, < \rangle \to (\omega + n)^{\omega+k}$ for all naturals $k \leq n$. For the most part, we cannot guarantee larger homogeneous sets than this:

Proposition (G.)

- $\langle \mathbb{R}, < \rangle \not\to (\omega + \omega)^\omega$
- $\langle \mathbb{R}, < \rangle \not\to (\omega + \omega)^{\omega+k}$ for $2 \leq k < \omega$

However:

Proposition (G., Schilhan)

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Higher analogues of the reals

We next looked at orders of the form $\langle {}^{\alpha}2, <_{\text{lex}} \rangle$ for α an arbitrary infinite ordinal. Write $\zeta = \omega^* + \omega$ for the order type of $\langle \mathbb{Z}, < \rangle$.

Proposition (Kruse?)

For all ordinals α,

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \not\to (\zeta)^3.$$

The same proof gives $\langle {}^{\alpha}2, <_{\text{lex}} \rangle \not\to (\zeta)^{\omega}$ and $\langle {}^{\alpha}2, <_{\text{lex}} \rangle \not\to (\zeta)^{\omega^*}$. If we increase the exponent further, however:

Proposition (Kruse?)

For all ordinals α,
$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \not\to (\zeta)^3.$$

Proposition (G., Weinert)

In ZF + AD, if $\alpha \geq \omega_1$,
$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\zeta)^{\zeta}.$$

This is a consequence of a more general result:

Theorem (G., Weinert)

In ZF + AD, if $\alpha \geq \omega_1$ and τ is any countable order type with $\omega\omega^* \not\leq \tau$ and $\omega^*\omega \not\leq \tau$,
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What happens if τ does embed one of $\omega\omega^*$ or $\omega^*\omega$?

Proposition (G.)

For all ordinals α, if τ is any of $\omega\omega^*$, $\omega^*\omega$, $\omega\zeta$, $\omega^*\zeta$, or ζ^2,

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \not\to (\tau)^{\tau}.$$

The colourings witnessing the failures of these relations are essentially all the same colouring, but this colouring does *not* give a failure of the relation for $\tau = \zeta\omega$ or $\tau = \zeta\omega^*$.

Question

Is it consistent that $\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\zeta\omega)^{\zeta\omega}$ (equivalently, that $\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\zeta\omega^*)^{\zeta\omega^*}$) for some α?

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Thank you!