Forcing and combinatorics of Van Douwen families

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Some combinatorial families of reals

Definition

A family $\mathcal{A}$ of infinite subsets of ω is called almost disjoint (ad) iff $A \cap B$ is finite for all $A \neq B \in \mathcal{A}$. $\mathcal{A}$ is called maximal almost disjoint (mad) iff its maximal with respect to inclusion.

In combinatorial set theory we want to understand the possible sizes of such maximal families of reals and their relation to the existence of maximal objects of other types.

Definition

We define the spectrum and cardinal characteristic of mad families:

$$\begin{aligned} \operatorname{spec}(\mathfrak{a}) &:= \{|\mathcal{A}| \mid \mathcal{A} \text{ is an infinite mad family}\} \\ \mathfrak{a} &:= \min(\operatorname{spec}(\mathfrak{a})) \end{aligned}$$

On mad families

Proposition

$\mathfrak{b} \leq \mathfrak{a}$, *where* $\mathfrak{b}$ *is the bounding number.*

Theorem (Shelah 2004, [9])

$\aleph_1 = \mathfrak{a} < \mathfrak{d}$ *in the Cohen model, where* $\mathfrak{d}$ *is the dominating number.*
$\aleph_1 < \mathfrak{d} < \mathfrak{a}$ *is consistent in a template model.*

Theorem (Hechler 1997, [5])

$\mathrm{spec}(\mathfrak{a})$ *is closed under singular limits.*

On eventually different families

Definition

Two functions $f, g : \omega \to \omega$ are called eventually different iff

$$\{n \in \omega \mid f(n) = g(n)\}$$

is finite. A family $\mathcal{F}$ of such functions is called eventually different (e.d.) iff all its members are pairwise eventually different. $\mathcal{F}$ is called maximal (m.e.d.) iff its maximal with respect to inclusion. Finally,

$$\mathrm{spec}(\mathfrak{a}_\mathrm{e}) := \{|\mathcal{F}| \mid \mathcal{F} \text{ is a m.e.d. family}\}$$
$$\mathfrak{a}_\mathrm{e} := \min(\mathrm{spec}(\mathfrak{a}_\mathrm{e}))$$

On eventually different families

Proposition (Bartoszynski, Judah 1995, [1])

$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_e$.

Theorem (Shelah 2004, [9]; Blass 2010, [2])

$\aleph_1 < \mathrm{non}(\mathcal{M}) < \mathfrak{a}_e$ *is consistent in a template model.*
$\aleph_1 = \mathfrak{a} < \mathrm{non}(\mathcal{M}) = \mathfrak{a}_e$ *holds in the random model.*

Questions for eventually different families

Question

Does ZFC *prove* $\mathfrak{a} \leq \mathfrak{a}_e$ *or is* $\mathfrak{a}_e < \mathfrak{a}$ *consistent?*

Question

Is $\mathrm{spec}(\mathfrak{a}_e)$ *closed under singular limits?*

Theorem (Brian 2021, [3])

$\mathrm{spec}(\mathfrak{a}_T)$ *is closed under singular limits.*

On Van Douwen families

Definition

Let $\mathcal{F}$ be an e.d. family. We say that $\mathcal{F}$ is a Van Douwen family iff for all infinite subsets $A \subseteq \omega$ also

$$\mathcal{F} \restriction A := \{f \restriction A \mid f \in \mathcal{F}\}$$

is a maximal eventually different family. In other words: For every partial function $g : A \to \omega$ there is a $f \in \mathcal{F}$ agreeing with g infinitely often.

Theorem (Zhang 1999, [10])

Under MA *there is a Van Douwen familiy of size continuum.*

Theorem (Raghavan 2010, [7])

There always is a Van Douwen family of size continuum.

On Van Douwen families

Definition

Hence, we may define the spectrum and cardinal characteristic:

$$\mathrm{spec}(\mathfrak{a}_\mathsf{v}) := \{|\mathcal{F}| \mid \mathcal{F} \text{ is a Van Douwen family}\}$$
$$\mathfrak{a}_\mathsf{v} := \min(\mathrm{spec}(\mathfrak{a}_\mathsf{v}))$$

Clearly, we have $\mathrm{spec}(\mathfrak{a}_\mathsf{v}) \subseteq \mathrm{spec}(\mathfrak{a}_\mathsf{e})$, so that $\mathfrak{a}_\mathsf{e} \leq \mathfrak{a}_\mathsf{v}$. However, we may answer the two previous questions for Van Douwen families!

Proposition (Folklore)

ZFC *proves that* $\mathfrak{a} \leq \mathfrak{a}_v$.

Theorem (S. 2024, [8])

$\mathrm{spec}(\mathfrak{a}_v)$ *is closed under singular limits.*

Van Douwen vs m.e.d. families

Question

Does $\mathfrak{a}_e = \mathfrak{a}_v$ *hold? Maybe even* $\operatorname{spec}(\mathfrak{a}_e) = \operatorname{spec}(\mathfrak{a}_v)$*?*

A positive answer would give us a positive answer for the two previous old questions for $\mathfrak{a}_\mathrm{e}$, respectively:

Question

Does $\mathfrak{a} \leq \mathfrak{a}_e$ *hold? Is* $\operatorname{spec}(\mathfrak{a}_e)$ *closed under singular limits?*

Measuring Van Douwen'ness

Towards the possible consistency of $\mathfrak{a}_e < \mathfrak{a}_v$ we may want to study maximal non Van Douwen families and their forcing indestructibility.

Definition (Raghavan)

Let $\mathcal{F}$ be a m.e.d. family. We define its associated ideal $\mathcal{I}_0(\mathcal{F})$ as

$$\mathcal{I}_0(\mathcal{F}) := \{A \in [\omega]^\omega \mid \mathcal{F} \restriction A \text{ is not a m.e.d. family}\} \cup \mathrm{Fin}\,.$$

Observation

$\mathcal{F}$ is a Van Douwen family iff $\mathcal{I}_0(\mathcal{F}) = \mathrm{Fin}$.

Definability of Van Douwen families

Theorem (Raghavan 2010, [7])

If $\mathcal{F}$ is analytic, then $\mathrm{Fin} \subsetneq \mathcal{I}_0(\mathcal{F})$.

Corollary

Van Douwen families cannot be analytic.

Theorem (Horowitz, Shelah 2016, [6])

There is a Borel maximal eventually different family.

Many non Van Douwen families

Can we realize every ideal as the associated ideal of some m.e.d. family? If $\mathcal{I}_0(\mathcal{F})$ is a maximal ideal we have for every $A \in [\omega]^\omega \setminus \mathrm{cofin}(\omega)$

1. either there is $g : A \to \omega$ such that $\mathcal{F} \restriction A \cup \{g\}$ is e.d.,
2. or there is $g : A^c \to \omega$ such that $\mathcal{F} \restriction A^c \cup \{g\}$ is e.d.,

so in some sense $\mathcal{F}$ would be as far as possible away from being Van Douwen.

Theorem (S. 2024, [8])

Assume CH. *Then for every ideal $\mathcal{I}$ containing* Fin *there is a m.e.d. such that $\mathcal{I}_0(\mathcal{F}) = \mathcal{I}$.*

Forcing indestructibility of the associated ideal

What about $\neg$CH? May the associated ideal exhibit some forcing indestructibility?

Definition

Let $\mathcal{F}$ be a m.e.d. family and $\mathbb{P}$ be a forcing. We say that $\mathcal{I}_0(\mathcal{F})$ is $\mathbb{P}$-indestructible iff for every $\mathbb{P}$-generic G in $V[G]$ we have that

$$\mathcal{I}_0(\mathcal{F})^{V[G]} = \langle \mathcal{I}_0(\mathcal{F})^V \rangle,$$

where $\langle \cdot \rangle$ is the generated ideal.

Note that the indestructibility of $\mathcal{I}_0(\mathcal{F})$ implies that the maximality of $\mathcal{F}$ is preserved.

Realizing Sacks-indestructible associated ideals

Theorem (S. 2024, [8])

Assume CH. *Then for every ideal* $\mathcal{I}$ *containing* Fin *there is a m.e.d.* $\mathcal{F}$ *such that* $\mathcal{I}_0(\mathcal{F}) = \mathcal{I}$ *and* $\mathcal{I}_0(\mathcal{F})$ *is* $\mathbb{S}^{\aleph_0}$*-indestructible.*

Theorem (Fischer, S. 2023, [4])

If $\mathcal{I}_0(\mathcal{F})$ *is* $\mathbb{S}^{\aleph_0}$*-indestructible, then* $\mathcal{I}_0(\mathcal{F})$ *is indestructible by any countably supported iteration or product of Sacks-forcing.*

Corollary (S. 2024, [8])

In the iterated Sacks-model for every $\aleph_1$*-generated ideal* $\mathcal{I}$ *there is a m.e.d. family* $\mathcal{F}$ *such that* $\mathcal{I} = \mathcal{I}_0(\mathcal{F})$*.*

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Thank you for your attention!