Productivity of selective covering properties

Marta Kładź-Duda

University of Warsaw, Faculty of Mathematics, Informatics and Mechanics

Hejnice 2025

joint work with Piotr Szewczak and Lyubomyr Zdomskyy

The research was funded by the National Science Center, Poland and the Austrian Science Found
under the Weave-UNISONO call in the Weave programme
Project: *Set-theoretic aspects of topological selections* 2021/03/Y/ST1/00122

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\quad$ $\mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\quad \mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $\quad X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O} \, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma$

Scheepers: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Omega$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma$

Scheepers: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Omega$

Menger: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{O}$

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\quad \mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $\quad X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma$

Scheepers: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Omega$

Menger: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{O}$

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Covers and covering properties

$\mathcal{O}$: all open covers of $X \subseteq 2^\omega$

Γ: all γ-cover of X $\quad \mathcal{U}$ is infinite and $\{ U \in \mathcal{U} : x \notin U \}$ is finite for all $x \in X$

Ω: all ω-covers of X $\quad X \notin \mathcal{U}$, for finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma$

Scheepers: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Omega$

Menger: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\, \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{O}$

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O} \; \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots \; \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\ \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots\ \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

$$U_n^m = \{x \in X : f(x)(n) = m\}$$

$$\begin{array}{cccccccc}
U_6^0 & U_6^1 & U_6^2 & U_6^3 & U_6^4 & U_6^5 & U_6^6 & U_6^7 \\
U_5^0 & U_5^1 & U_5^2 & U_5^3 & U_5^4 & U_5^5 & U_5^6 & U_5^7 \\
U_4^0 & U_4^1 & U_4^2 & U_4^3 & U_4^4 & U_4^5 & U_4^6 & U_4^7 \\
U_3^0 & U_3^1 & U_3^2 & U_3^3 & U_3^4 & U_3^5 & U_3^6 & U_3^7 \\
U_2^0 & U_2^1 & U_2^2 & U_2^3 & U_2^4 & U_2^5 & U_2^6 & U_2^7 \\
U_1^0 & U_1^1 & U_1^2 & U_1^3 & U_1^4 & U_1^5 & U_1^6 & U_1^7 \\
U_0^0 & U_0^1 & U_0^2 & U_0^3 & U_0^4 & U_0^5 & U_0^6 & U_0^7 \\
\mathcal{U}_0 & \mathcal{U}_1 & \mathcal{U}_2 & \mathcal{U}_3 & \mathcal{U}_4 & \mathcal{U}_5 & \mathcal{U}_6 & \mathcal{U}_7
\end{array}$$

Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\ \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots\ \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

$$U_n^m = \{x \in X : f(x)(n) = m\}$$

U_6^0	U_6^1	U_6^2	U_6^3	U_6^4	U_6^5	U_6^6	U_6^7
U_5^0	U_5^1	U_5^2	U_5^3	U_5^4	U_5^5	U_5^6	**U_5^7**
U_4^0	U_4^1	U_4^2	**U_4^3**	U_4^4	**U_4^5**	**U_4^6**	**U_4^7**
U_3^0	U_3^1	**U_3^2**	**U_3^3**	U_3^4	**U_3^5**	**U_3^6**	**U_3^7**
U_2^0	U_2^1	**U_2^2**	**U_2^3**	**U_2^4**	**U_2^5**	**U_2^6**	**U_2^7**
U_1^0	**U_1^1**	**U_1^2**	**U_1^3**	**U_1^4**	**U_1^5**	**U_1^6**	**U_1^7**
U_0^0	**U_0^1**	**U_0^2**	**U_0^3**	**U_0^4**	**U_0^5**	**U_0^6**	**U_0^7**
$\mathcal{U}_0$	$\mathcal{U}_1$	$\mathcal{U}_2$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_5$	$\mathcal{U}_6$	$\mathcal{U}_7$

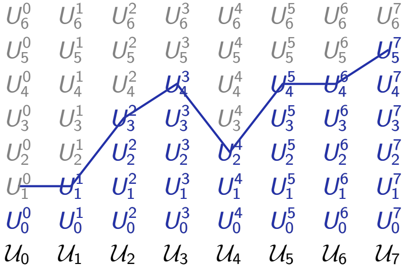
Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\ \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots\ \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

$$U_n^m = \{x \in X : f(x)(n) = m\}$$



$$\gamma \to g$$

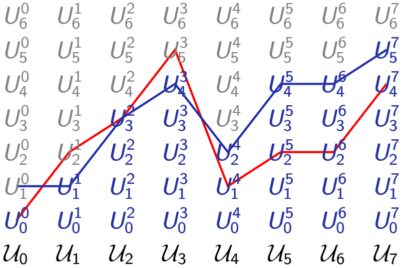
Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\ \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots\ \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

$$U_n^m = \{x \in X : f(x)(n) = m\}$$



$$x \mapsto f_x \leq^* g$$

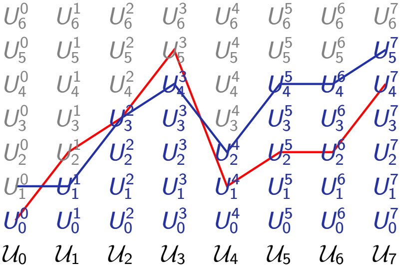
Hurewicz-Recław Theorem

Hurewicz: $\forall \mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{O}\ \exists$ finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots\ \{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$

Theorem (Hurewicz-Recław)

X is Hurewicz $\Leftrightarrow$ every continuous image of X in ω^ω is bounded.

$$U_n^m = \{x \in X : f(x)(n) = m\}$$



Theorem (Hurewicz-Recław)

- *X is Menger $\Leftrightarrow$ no continuous image of X in ω^ω is dominaiting.*

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

U : free ultrafilter

$f \leq_U g \Leftrightarrow \{n : f(n) \leq g(n)\} \in U$

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

U : free ultrafilter

$f \leq_U g \Leftrightarrow \{n : f(n) \leq g(n)\} \in U$

$\mathfrak{b}(U)$: the minimal size of a $\leq_U$-unbounded set

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

U : free ultrafilter

$f \leq_U g \Leftrightarrow \{n : f(n) \leq g(n)\} \in U$

$\mathfrak{b}(U)$: the minimal size of a $\leq_U$-unbounded set

dominaiting scale $S = \{f_\alpha : \alpha < \mathfrak{d}\} \in \omega^\omega$ $\alpha < \beta \Rightarrow f_\alpha \leq^* f_\beta$ S is dominaiting

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

U : free ultrafilter

$f \leq_U g \Leftrightarrow \{n : f(n) \leq g(n)\} \in U$

$\mathfrak{b}(U)$: the minimal size of a $\leq_U$-unbounded set

dominaiting scale $S = \{f_\alpha : \alpha < \mathfrak{d}\} \in \omega^\omega$ $\alpha < \beta \Rightarrow f_\alpha \leq^* f_\beta$ S is dominaiting

$\mathfrak{b}(U)$-scale $S = \{f_\alpha : \alpha < \mathfrak{b}(U)\} \in \omega^\omega$ $\alpha < \beta \Rightarrow f_\alpha \leq_U f_\beta$ S is $\leq_U$-unbounded

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

$\mathfrak{d}$-unbounded set: X such that $|X| \geq \mathfrak{d}$ and $Y \subseteq X$ bounded $\Rightarrow |Y| < \mathfrak{d}$

Theorem

Let $X \subseteq [\omega]^\omega$ be a $\mathfrak{d}$-unbounded set. Then $X \cup \mathrm{Fin}$ is Menger.

Theorem

Assume $\mathfrak{b} = \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be an unbounded set. Then it has a $\mathfrak{d}$-unbounded subset.

U : free ultrafilter

$f \leq_U g \Leftrightarrow \{n : f(n) \leq g(n)\} \in U$

$\mathfrak{b}(U)$: the minimal size of a $\leq_U$-unbounded set

dominaiting scale $S = \{f_\alpha : \alpha < \mathfrak{d}\} \in \omega^\omega$ $\alpha < \beta \Rightarrow f_\alpha \leq^* f_\beta$ S is dominaiting

$\mathfrak{b}(U)$-scale $S = \{f_\alpha : \alpha < \mathfrak{b}(U)\} \in \omega^\omega$ $\alpha < \beta \Rightarrow f_\alpha \leq_U f_\beta$ S is $\leq_U$-unbounded

productively Hurewicz: X such that for every Hurewicz set H product $X \times H$ is Hurewicz

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there is a* $\mathfrak{d}$*-unbounded set* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there is a* $\mathfrak{d}$*-unbounded set* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem (Canjar)

$\mathrm{cf}(\mathfrak{d}) = \mathfrak{d} \Rightarrow$ *There is an ultrafilter* U *such that* $\mathfrak{b}(U) = \mathfrak{d}$*.*

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there is a* $\mathfrak{d}$*-unbounded set* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem (Canjar)

$\mathrm{cf}(\mathfrak{d}) = \mathfrak{d} \Rightarrow$ *There is an ultrafilter* U *such that* $\mathfrak{b}(U) = \mathfrak{d}$.

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathrm{cf}\,\mathfrak{d} = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there are an ultrafilter* U *and a* $\mathfrak{b}(U)$*-scale* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

$\mathfrak{d}$-unbounded sets and $\mathfrak{b}(U)$-scales

σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there is a* $\mathfrak{d}$*-unbounded set* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem (Canjar)

$\mathrm{cf}(\mathfrak{d}) = \mathfrak{d} \Rightarrow$ *There is an ultrafilter* U *such that* $\mathfrak{b}(U) = \mathfrak{d}$*.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathrm{cf}\,\mathfrak{d} = \mathfrak{d}$ *and* X *contain a* $\mathfrak{d}$*-unbounded set. Then there are an ultrafilter* U *and a* $\mathfrak{b}(U)$*-scale* Y *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem (Szewczak-Tsaban)

Let U *be an ultrafilter,* X *is Hurewicz and* Y *is a* $\mathfrak{b}(U)$*-scale. Then* $X \times (Y \cup \mathrm{Fin})$ *is Scheepers.*

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

$Z \subset [\omega]^{\omega}$ – unbounded continuous image of $X \times H$

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

$Z \subset [\omega]^\omega$ – unbounded continuous image of $X \times H$

Y – $\mathfrak{b}(U)$-scale such that $Z \times (Y \cup \mathrm{Fin})$ is not Menger

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$*. Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$*. Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

$Z \subset [\omega]^\omega$ – unbounded continuous image of $X \times H$

Y – $\mathfrak{b}(U)$-scale such that $Z \times (Y \cup \mathrm{Fin})$ is not Menger

$X \times H \times (Y \cup \mathrm{Fin}) \xrightarrow{\text{continuous}} Z \times (Y \cup \mathrm{Fin}) \Rightarrow X \times H \times (Y \cup \mathrm{Fin})$ is not Menger

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

$Z \subset [\omega]^\omega$ – unbounded continuous image of $X \times H$

Y – $\mathfrak{b}(U)$-scale such that $Z \times (Y \cup \mathrm{Fin})$ is not Menger

$X \times H \times (Y \cup \mathrm{Fin}) \xrightarrow{\text{continuous}} Z \times (Y \cup \mathrm{Fin}) \Rightarrow X \times H \times (Y \cup \mathrm{Fin})$ is not Menger

$H \times (Y \cup \mathrm{Fin})$ is Scheepers $\Rightarrow X \times H \times (Y \cup \mathrm{Fin})$ is Scheepers

Result

Theorem (Szewczak-Tsaban)

Let $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. *Then every productively Menger space is productively Hurewicz.*

Theorem (Szewczak-KD-Zdomskyy)

Let $\mathfrak{b} = \mathfrak{d}$. *Then every productively Scheepers space is productively Hurewicz.*

Sketch of the proof:

Assume X is productively Scheepers, H is Hurewicz and $X \times H$ isn't Hurewicz.

$Z \subset [\omega]^\omega$ – unbounded continuous image of $X \times H$

Y – $\mathfrak{b}(U)$-scale such that $Z \times (Y \cup \mathrm{Fin})$ is not Menger

$X \times H \times (Y \cup \mathrm{Fin}) \xrightarrow{\text{continuous}} Z \times (Y \cup \mathrm{Fin}) \Rightarrow X \times H \times (Y \cup \mathrm{Fin})$ is not Menger

$H \times (Y \cup \mathrm{Fin})$ is Scheepers $\Rightarrow X \times H \times (Y \cup \mathrm{Fin})$ is Scheepers

Scheepers implies Menger – contradiction $\square$