Inaccessible Cardinals without Choice

Hope Duncan

January 2025

What I hope you'll take away from this talk

Today's talk focuses on how we best translate the notion of a cardinal being inaccessible into a choiceless context. This is an interesting question in its own right, but has fascinating consequences when attempting to define large cardinal notions of a higher consistency strength in ZF.

This talk will cover:

- Considerations we need when defining inaccessible cardinals in ZF.
- The definitions of various types inaccessible cardinals in ZF and the relationships between them.
- Potential directions for research into choiceless large cardinals.

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The definition of an inaccessible cardinal in ZFC

Definition

(ZFC) A cardinal κ is inaccessible if it is a regular strong limit cardinal.

In ZFC, by strong limit cardinal we mean that for all $\lambda < \kappa$, we have $2^\lambda < \kappa$.

The immediate issue we have here is that for well-orderable λ, 2^λ may not be well-orderable!

Recall: An equivalence of AC is the cardinal trichotomy - for a given two sets, they either have the same cardinality, or one has smaller cardinality than the other.

Let us consider several notions of one set being 'smaller than' another in ZF:

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Comparing Sets

In ZFC, we frequently use the notation $X \leq Y$ to refer to the equivalent notions that 'there is an injection from X to Y' and (assuming X is non empty) that 'there is a surjection from Y to X'. These are no longer equivalent over models of ZF, and we may define 3 distinct definitions of a set being smaller than some fixed set.

- $X <_i \kappa$ is there is some $\alpha < \kappa$ and an injection from X into α.
- $X <_{\bar{\imath}} Y$ if there is no injection from Y into X.
- $X <_{\bar{s}} Y$ if there is no surjection from X onto Y.

Note that it is equivalent over ZF to phrase the first definition as there is some $\alpha < \kappa$ and a surjection from α onto X. Over ZFC (or if the sets we're considering are well orderable), all of the above definitions coincide.

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Strong Limits in ZF

Using these definitions of 'less than' we can give several non-equivalent statements of what it means to be a strong limit and therefore non-equivalent definitions of inaccessibility.

Definition

We say a limit cardinal κ is an $x-$strong limit cardinal if for all $\lambda < \kappa$, we have $2^\lambda <_x \kappa$.
Assume κ is uncountable.

- κ is i-inaccessible if it is a regular $i-$strong limit cardinal
- κ is $\bar{s}$-inaccessible if it is a regular $\bar{s}-$strong limit cardinal
- κ is $\bar{\imath}$-inaccessible if it is a regular $\bar{\imath}-$strong limit cardinal

Recall we can also define a notion of inaccessiblity using s-strong limit cardinals, but as this is equivalent to define an i-strong inaccessible we will not mention this going forward for brevity.

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How should be choose a definition in ZF?

An important consideration when trying to somehow 'translate' a ZFC definition into ZF is thinking about the important mathematical concepts that the original definition captures.

A standard theorem about inaccessibile cardinals is that κ is inaccessible if and only if V_κ is a model of second order ZFC. We can do the same in ZF.

Definition

κ is honestly inaccessible if V_κ is a model of ZF_2, second order ZF.

Note you will see this referred to in the literature as a '$v-$inaccessible' cardinal'.

An interesting equivalence to note is that κ is honestly inaccessible if and only if there is no such $x \in V_\kappa$ such that $f : x \to \kappa$ is cofinal.

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Relationship between notions of inaccessibility

We have a series of (one way!) implications between the different forms of inaccessible cardinals.

Theorem

κ is i-inaccessible $\Rightarrow$ κ is honestly inaccessible $\Rightarrow$ κ is $\bar{\mathfrak{s}}$-inaccessible $\Rightarrow$ κ is $\bar{\imath}$-inaccessible.

It is also interesting at this stage to note just how strong being *i*-inaccessible is in a ZF context - specifically that if there exists an uncountable *i*-strong limit, then there is a well-ordering of the real numbers.

$X <_i \alpha$ for an ordinal α implies that X is well-orderable. So if κ is uncountable, then $\aleph_0 < \kappa$, so $\mathcal{P}(\omega)$ is well-orderable.

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Sketch Proofs

1. κ *i*-inaccessible $\Rightarrow$ κ honestly inaccessible: we can show that for $\alpha < \kappa$, every V_α is well-orderable and has cardinality $< \kappa$ (in particular, we cannot map V_α for $\alpha < \kappa$ onto κ).
2. κ honestly inaccessible $\Rightarrow$ κ $\bar{\mathfrak{s}}$-inaccessible: κ being honestly inaccessible is equivalent to the statement that κ is a regular, uncountable cardinal and for all $\alpha < \kappa$, we have $V_\alpha <_{\bar{\mathfrak{s}}} \kappa$. Notice that $\mathcal{P}(\lambda) \subseteq V_{\lambda+1}$ and the result follows.
3. κ $\bar{\mathfrak{s}}$-inaccessible $\Rightarrow$ κ $\bar{\imath}$-inaccessible: There is no surjection from X onto Y implies there is no injection from Y into X, the result immediately follows.

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These implications are one way

- The proofs which show that these implications are all one-way are a little involved, so I won't give explicit proofs, rather give a brief overview of the technique used, symmetric extensions.
- When taking a forcing extension, $V \subseteq V[G]$, we may consider an intermediate model, $V \subseteq M \subseteq V[G]$, where M is our 'symmetric extension', a model which need not satisfy AC.
- To generate M, we take a symmetric system $\mathcal{S}$, a triple made of our forcing poset $\mathbb{P}$, a group of automorphisms $\mathcal{G}$, and a filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. Our model is generated by the names which are hereditarily 'symmetric enough', i.e. the group of automorphisms which fix the name is in the filter.
- To show that e.g. κ is i-inaccessible $\Rightarrow$ κ is honestly inaccessible must be one way, we find a symmetric extension in which there exists κ which is honestly inaccessible, but not i-inaccessible.

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Equiconsistency!

It is interesting to note however, that the existence of a standard ZFC inaccessible cardinal is equiconsistent to the existance of any of these ZF formulations of inaccessibility!

Theorem

All our formulations of inaccessibility are equiconsistent.

Proof.

Con(ZFC + $\exists$ an inaccessible) $\Rightarrow$ Con(ZF + $\exists$ an i-inaccessible) $\Rightarrow$ Con(ZF + $\exists$ an honestly inaccessible) $\Rightarrow$ Con(ZF + $\exists$ an $\bar{\mathfrak{s}}$-inaccessible) $\Rightarrow$ Con(ZF + $\exists$ an $\bar{\imath}$-inaccessible) is immediate. To finish, we will show that if κ is $\bar{\imath}$-inaccessible, then κ is inaccessible in L. It must hold that κ is regular in L. Let $\alpha < \kappa$. If $L \models$ "There is an injection from κ into $\mathcal{P}(\alpha)$",
then there is an injection from κ into $\mathcal{P}^L(\alpha) \subseteq \mathcal{P}(\alpha)$, so$\kappa$ is not $\bar{\imath}$-inaccessible in the universe. □

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Higher consistency strength large cardinals in ZF

- Other large cardinals, notably weakly compact cardinals may defined in relation to inaccessible cardinals.
- Recall the 'original' definition of κ being weakly compact: for any collection of $\mathcal{L}_{\kappa,\kappa}$ sentences using at most κ many non-logical symbols, if every sub-collection of cardinality $< \kappa$ has a model, then the whole collection has a model.
- So, which once equivalent formulations of weak compactness over ZFC are still equivalent over ZF? What is the 'correct' definition of inaccessibility to use?
- Consider κ is weakly compact if $\kappa \to (\kappa)^2_2$ for $\kappa > \omega$. In ZF, ω_1 can be weakly compact using this definition, a huge failure of choice! When we examine the proofs of equivalences between this definition and other formulations of weak compactness, choice is used heavily, so this is not too surprising, but for different formulations of weak compactness this is not necessarily the case.

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Elementary embeddings in ZF definition

Often definitions based on elementary embeddings will translate well to being used in a ZF context.

A good example of this is defining weakly critical cardinals.

Definition

κ is a weakly critical cardinal if for every $A \subseteq V_\kappa$, there exists an elementary embedding $j : X \to M$ with critical point κ, where X and M are transitive, and $\kappa, V_\kappa, A \in X \cap M$.

Assuming choice, this definition is equivalent to κ being weakly compact. A standard ZFC equivalence to weak compactness is the *extension property*, κ has the extension property if for every $A \subseteq V_\kappa$, there is a transitive elementary end extension of $\langle V_\kappa, \in, A\rangle$.

In ZF that κ is weakly critical if and only if κ has the extension property. Also if κ is weakly critical, then κ must be honestly inaccessible.

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Often definitions based on elementary embeddings will translate well to being used in a ZF context.
A good example of this is defining weakly critical cardinals.

Definition

κ is a weakly critical cardinal if for every $A \subseteq V_\kappa$, there exists an elementary embedding $j : X \to M$ with critical point κ, where X and M are transitive, and $\kappa, V_\kappa, A \in X \cap M$.

Assuming choice, this definition is equivalent to κ being weakly compact. A standard ZFC equivalence to weak compactness is the *extension property*, κ has the extension property if for every $A \subseteq V_\kappa$, there is a transitive elementary end extension of $\langle V_\kappa, \in, A\rangle$.
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Future research directions

- My current research aims to define a tree property which captures the notion of weak compactness in the ZF context, along with the assumption of honest inaccessibility. It will be interesting to see how this interacts with other ZFC equivalences of weak compactness, or higher consistency strength large cardinals. Work to study the relationship between other ZFC formulations of weak compactness in ZF is currently being done by myself and Lyra Gardiner.
- An interesting 'lighthouse problem' to work towards in choiceless large cardinal research is the consistency strength of a particular definition of a supercompact cardinal in ZF, as it is currently open whether we can get a non-trivial failure of choice in this context without assuming a Reinhardt.

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Supercompacts in ZF

Definition

(Woodin)(ZF)A cardinal κ is supercompact if for every $\alpha > \kappa$, there exists $\beta > \alpha$ and an elementary embedding, $j : V_\beta \to N$ such that:

- N is a transitive set and $N^{V_\alpha} \subseteq N$,
- j has a critical point κ, and $\alpha < j(\kappa)$.

There is also a theorem of Woodin which states that if we have a supercompact, we may force DC to hold.
We can start with a model of ZFC + 'there exists a supercompact', use e.g. the Cohen model to break choice, then force DC. However in doing so we will have forced back the whole of AC, as the failure of choice comes from a 'seed' inside V_κ. So can we violate choice in a non-trivial way, so that we don't force the whole of choice back (i.e. SVC fails) but there is still a supercompact in the universe?

Thanks for listening!

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