Big Ramsey degrees — current status and open problems

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Prague

Winter school, 2025, Hejnice

Advertisement

1. J.H., M. Konečný.
   Twenty years of Nešetřil's classification programme of Ramsey classes.
   https://arxiv.org/abs/2501.17293, January 28 2025, 65pp
2. J.H., A. Zucker.
   A survey on big Ramsey structures.
   https://arxiv.org/abs/2407.17958, July 2025, 34pp

Non-strucutral Ramsey results

Finite pigeonhole principle

$$\forall_{n,r>0}\exists_{N>0} : N \longrightarrow (n)^{\bullet}_{r}.$$

Infinite pigeonhole principle

$$\forall_{r>0} : \omega \longrightarrow (\omega)^{\bullet}_{r}.$$

Theorem (Finite Ramsey theorem, 1930)

$$\forall_{n,k,r>0}\exists_{N>0} : N \longrightarrow (n)^{k}_{r}.$$

Definition (Erdős–Rado partition arrow)

$N \longrightarrow (n)^k_r$ means:
For every partition of $\binom{N}{k}$ into r classes (colours) there exists $X \in \binom{N}{n}$ such that $\binom{X}{k}$ belongs to a single part.

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Coloring infinite sets: Galvin–Přírký, Silver, Ellentuck. . .

Dual Non-structural Ramsey results

1. **Finite pigeonhle**: Hales–Jewett theorem 1963
2. **Infinite pigeonhole**: Infinite Hales–Jewett theorem (Carlson–Simpson Lemma 1984, independently obtained by Voigt)
3. **Finite Ramsey theorem**: Graham–Rothschild Theorem, 1971
4. **Infinite Ramsey theorem**: Carlson–Simpson 1984.

Eventually generalized to an axiomatic approach to Ramsey spaces by Todorcevic.

Finite structural pigeonhole

Let $\mathcal{G}$ be the class of all graphs.

Theorem (Finite structural pigeonhole: Folkman 1970)

$$\forall_{\mathbf{G}\in\mathcal{G}}\forall_{r>0}\exists_{\mathbf{H}\in\mathcal{G}} : \mathbf{H} \longrightarrow (\mathbf{G})^{\bullet}_{r}.$$

For every finite graph **G** and every $r > 0$ there exists finite graph **H** such that every r-coloring of vertices of **H** contains induced monochromatic subgraph isomorphic to **G**.

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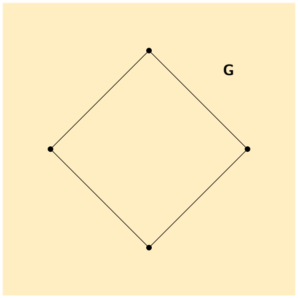

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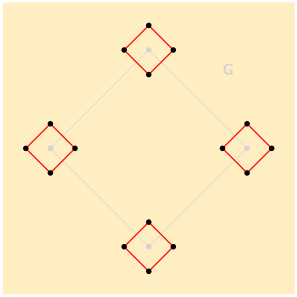

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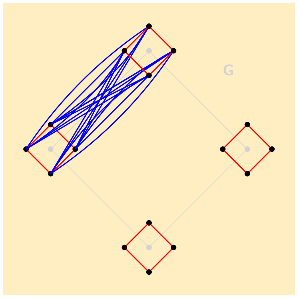

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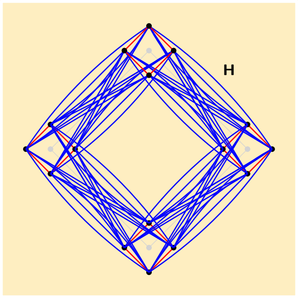

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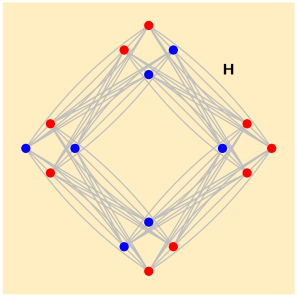

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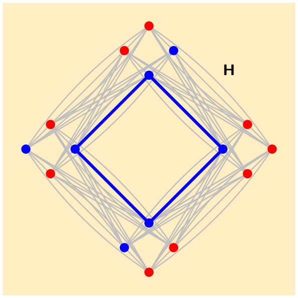

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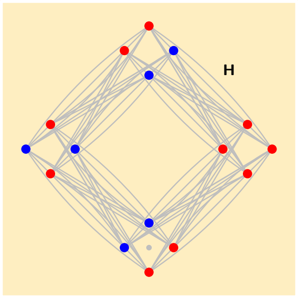

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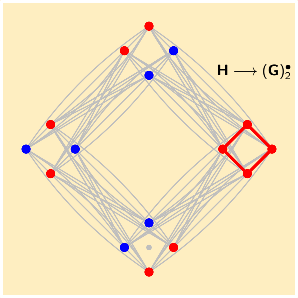

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Let $\mathcal{G}$ be the class of all finite graphs.

Theorem (Restricted finite structural pigeonhole: Folkman 1970)

$$\forall_{\mathbf{G}\in\mathcal{G}}\forall_{r>0}\exists_{\mathbf{H}\in\mathcal{G}} : \mathbf{H} \longrightarrow (\mathbf{G})^{\bullet}_{r}.$$

Moreover the size of largest clique of **H** *equals to the size of largest clique of* **G**.

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For every $u > 1$ and $r, g > 0$ there exists an u-uniform hypergraph of girth at least g and chromatic number at least r.

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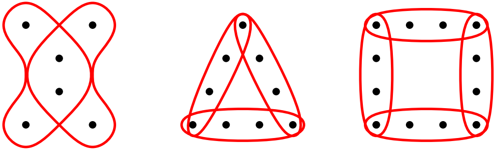
cychle of length ℓ in a hypergraph **H** is a sequence $v_0, E_0, v_1, E_1, \ldots v_{\ell-1}, E_{\ell-1}$ of distinct vertices and edges such that for every $i < \ell - 1$ it holds that $v_i, v_{i+1} \in E_i$ and $v_0 \in E_{\ell-1}$.
girth is the length of the shortest cycle in a hypergraph.



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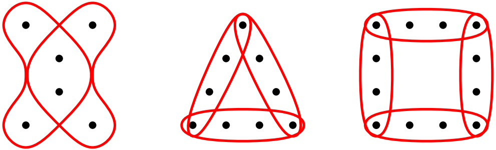

Erdős–Hajnal $\Longrightarrow$ Folkman.

Given $\mathbf{G}$ and r obtain $|\mathbf{G}|$-uniform hypergraph of girth 4 and the chromatic number $r+1$ and replace every hyper-edge by a copy of $\mathbf{G}$. $\square$

Infinite structural pigeonhole

Theorem (Infinite structural pigeonhole)

Let $\mathbf{R}$ *be the Rado (countable universal and homogeneous) graph.*

$$\mathbf{R} \longrightarrow (\mathbf{R})^{\bullet}_{2}.$$

Also true for any other countable universal graph.

Theorem (Restricted infinite structural pigeonhole: Komjáth–Rödl 1983)

Let $\mathbf{R}_3$ *be the (countable) homogeneous universal triangle-free graph.*

$$\mathbf{R}_3 \longrightarrow (\mathbf{R}_3)^{\bullet}_{2}.$$

Theorem (Restricted infinite structural pigeonhole: El-Zahar–Sauer, 1989)

Let $k \geq 3$ *and* $\mathbf{R}_k$ *be the (countable) homogeneous universal* $\mathbf{K}_k$*-free graph.*

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Theorem (Nguyen Van Thé–Sauer, 2009)

Let d be a finite integer and $\mathbf{M}_d$ the (countable) homogeneous universal metric space with integer distances at most d.

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New, simple, proof (also for ℓ_∞) appeared on arXiv today! (Bice, de Rancourt, J.H., Konečný)

More known examples: hypergraphs, posets, certain free amalgamation classes,. . .

Open problems

As the most general result, Zucker in 2020 characterised finitely constrained free amalgamation classes in binary languages whose Fraïssé limits satisfies infinite structural pigeonhole.

Sauer's unpublished results treat some infinitely constrained free amalgamation classes in binary language.

Many cases are open, including:

1. More general oscillation stability framework
2. Characterisation of Fraïssé limits of free amalgamation classes with relations of arity greater than 2
3. ...

Towards finite structural Ramsey theorem

Let $\mathcal{G}_k$ be the class of all finite graphs without clique of size k.

Theorem (Nešetřil–Rödl, 1975)

For every finite triangle-free graph $\mathbf{G}$ *there exists a finite triangle-free graph* $\mathbf{H}$ *such that for every* 2*-colouring of the edges of* $\mathbf{H}$ *there exists a monochromatic copy of* $\mathbf{G}$*:*

$$\forall_{\mathbf{G}\in\mathcal{G}_3}\exists_{\mathbf{H}\in\mathcal{G}_3} : \mathbf{H} \longrightarrow (\mathbf{G})^{\bullet\!-\!\bullet}_2$$

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Theorem (Nešetřil–Rödl, 1976)

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Proof techniques are different, due to the lack of an edge-coloring version of Erdős–Hajnal theorem. Origin of the partite construction.

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138 pages, 27 beautiful figures

Towards infinite structural Ramsey theorem

Theorem (Erdős–Hajnal–Pósa, 1973)

There exists a persistent 2-coloring of edges of the Rado graph $\mathbf{R}$ *where every copy of* $\mathbf{R}$ *contains both colours.*

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Theorem (Sauer–Pouzet 1996; also follows from unpublished Laver's proof)

Let **R** *be a Rado graph and let edges of* **R** *be finitely coloured. Then there is induced copy of* **R** *with at most 2 colors:* $\forall_{r>1} : \mathbf{R} \longrightarrow (\mathbf{R})^{\bullet\!-\!\bullet}_{r,2}$.

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Theorem (Balko–Chodounský–Dobrinen–Hubička–Konečný–Vena–Zucker, 2024)

The big Ramsey degree of non-edge in $\mathbf{R}_3$ *is 5.*

A special case of a general result for finitely constrained free amalgamation classes in finite binary languages.

Finite structural Ramsey theorem

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

Definition (Leeb's generalization of the Erdős–Rado partition arrow)

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$ means:

For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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Class **K** of structures is Ramsey if

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{K}} \exists_{\mathbf{C}\in\mathcal{K}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2,1}.$$

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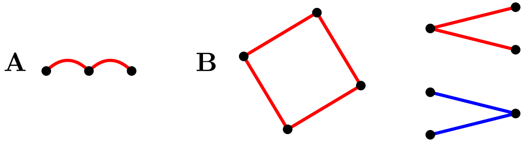

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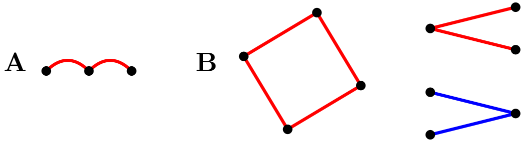

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Theorem (Nešetřil–Rödl 1977, Abramsohn–Harrington 1978)

Let L be a relational language containing a binary relation $<$, and let $\mathcal{K}$ be the class of all finite ordered L-structures. Then

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Theorem (Ordering property, Nešeřil–Rödl 1978)

Let **A** *be an L-structure, then there exist* **B** *such that every ordering of vertices of* **B** *contains all possible orderings of* **A** *as substructures.*

Finite structural Ramsey theorem

Theorem (Nešetřil–Rödl 1977, Abramsohn–Harrington 1978)

Let L be a relational language containing a binary relation $<$, and let $\mathcal{K}$ be the class of all finite ordered L-structures. Then

$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{K}}\exists_{\mathbf{C}\in\mathcal{K}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Theorem (Ordering property, Nešeřil–Rödl 1978)

*Let **A** be an L-structure, then there exist **B** such that every ordering of vertices of **B** contains all possible orderings of **A** as substructures.*

Theorem (Nešetřil–Rödl theorem 1977)

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$$\forall_{\mathbf{A},\mathbf{B}\in\mathcal{K}}\exists_{\mathbf{C}\in\mathcal{K}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

*Moreover, **C** can be constructed in a way so that every irreducible substructure of **C** (without the order) is contained in a copy of **B**.*

Finite structural Ramsey theorem

Examples of known Ramsey classes

- linear orders (Ramsey 1930)
- boolean algebras with anti lexicographic ordering (Graham–Rothschild 1972)
- linearly ordered free amalgamation classes (Nešetřil–Rödl theorem, 1977)
- partial orders with linear extensions (Nešetřil–Rödl, 1984, proofs later)
- dual relational structures with anti lexicographic ordering
  (Prömel 1985, Frankl, Rödl, Graham 1987)

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- Metric spaces (Nešetřil, 2005)
- Local dense order with unary predicate (Laflamme, Nguyen Van Thé, Sauer 2010)
- Linearly ordered partial Steiner system (Bhat–Nešetřil–Reiher–Rödl 2018)
- Linearly ordered bowtie-free graphs (Hubička–Nešetřil 2018)
- . . .

Finite structural Ramsey theorem

Theorem (Hubička–Nešetřil 2019)

Let L be a ~~relational~~ language containing a binary relation $<$, and let $\mathcal{K}$ be the class of all finite ordered L-structures. Then $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{K}}\exists_{\mathbf{C}\in\mathcal{K}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

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Definition (Homomorphism-embedding)

Let $\mathbf{A}$ and $\mathbf{B}$ be structures. A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism-embedding if the restriction $f|_{\mathbf{C}}$ is an embedding whenever $\mathbf{C}$ is an irreducible substructure of $\mathbf{A}$.

Theorem (Blackbox on the partite construction, Hubička–Nešetřil 2019)

Let L be a language, $n \geq 1$, and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}_0$ finite L-structures such that $\mathbf{A}$ is irreducible and $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. Then there exists a finite L-structure $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ and

1. *there exists a homomorphism-embedding $\mathbf{C} \to \mathbf{C}_0$,*
2. *for every substructure $\mathbf{C}'$ of $\mathbf{C}$ with at most n vertices there exists a structure $\mathbf{T}$ which is a tree amalgam of copies of $\mathbf{B}$, and a homomorphism-embedding $\mathbf{C}' \to \mathbf{T}$, and*
3. *every irreducible substructure of $\mathbf{C}$ extends to a copy of $\mathbf{B}$.*

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Open problem: Can we strenghten 2 so $\mathbf{C}'$ embedds into $\mathbf{T}$?

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Question (Bodirsky–Pinsker–Tsankov 2013)

Does the age of every ω-categorical structure have a precompact Ramsey expansion?

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Some explicit open cases:

1. Ramsey expansion of the class of all finite graphs of girth 4
2. Ramsey expansion of the class of all hypergraphs omitting odd cycles up to given length ℓ.
3. Ramsey expansion of H_4-free tournaments
4. Ramsey expansion of the class of all finite groups
5. Are equipartitions Ramsey?

Infinite structural Ramsey theorem, arity ≤ 2

Theorem (Upper bound by Laver 1969, characterisation by Devlin 1979)

The order of rationals $(\mathbb{Q}, \leq)$ has finite big Ramsey degrees: for every $n \in \omega$ there exists $T(n) \in \omega$ such that whenever n-element subsets of $\mathbb{Q}$ are finitely colored, there exists a copy of $(\mathbb{Q}, \leq)$ in itself touching at most $T(n)$ many colors.

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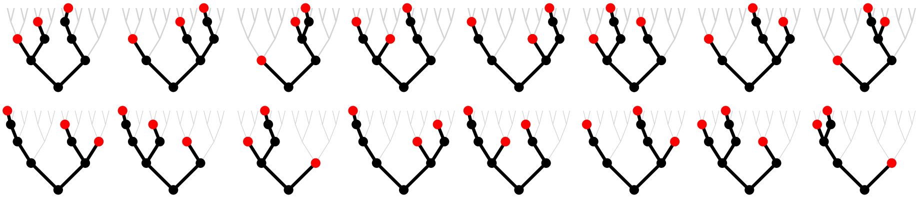

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Proof technique, based on the Milliken tree theorem generalizes to other cases:

1. Laflamme, Sauer, Vuksanovic (2006): Characterisation of big Ramsey degrees of Rado graph.
2. Nguyen Van Thé (2009): Characterisation of big Ramsey degrees of homogeneous ultrametric spaces.
3. Laflamme, Nguyen Van Thé, Sauer (2010): Characterisation of big Ramsey degrees of homogeneous dense local order.

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Big Ramsey degrees of the countable universal homogeneous triangle-free graphs are finite.

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Let L be a language with only finitely many relations of every arity > 1*. Then the Fraïssé limit of all finite L-structures where all relations are injective has finite big Ramsey degrees.*

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Open problems

1. Let **P** be a 3-uniform hypergraph on 4 vertices with all but one hyper-edge. Does the universal and homogeneous **P**-free hypergraph admit upper bounds on big Ramsey degrees?
2. What is the big Ramsey degree behaviour of boolean algebras
3. Identify more classes with type-respecting amalgamation property.
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Definition (Big Ramsey structure, Zucker 2017)

Let $\mathbf{K}$ be a countable structure and let $\mathbf{K}^*$ be an expansion of $\mathbf{K}$. We call $\mathbf{K}^*$ a *big Ramsey structure* for $\mathbf{K}$ if the following holds:

1. The colouring of $\mathbf{K}$ given by $\mathbf{K}^*$ is unavoidable. Namely, for every finite substructure $\mathbf{A}^*$ of $\mathbf{K}^*$ and every embedding $f\colon \mathbf{K} \to \mathbf{K}$ it holds that there is an embedding $e\colon \mathbf{A}^* \to \mathbf{K}^*$ such that $e[A] \subseteq f[K]$.
2. For every finite substructure $\mathbf{A}$ of $\mathbf{K}$, the number of its mutually non-isomorphic expansions in $\mathbf{K}^*$ is finite and equal to the big Ramsey degree of $\mathbf{A}$ in $\mathbf{K}$.

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