Generalized Haar meager sets and related cardinal invariants

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Let G be a non-locally compact group. PROBLEM: there is no Haar measure on the group G.

But we like (Haar) measure zero sets (don't we?).

Definition (Christensen, 1972, variation I)

Let G be a group and $A \subseteq G$. The set A is called a Haar null set if there exists a probability measure μ and a Borel set B such that $A \subseteq B$ and $\mu(gBh) = 0$ for every $g, h \in G$. Notation: $\mathcal{HN}(G)$.

Definition (Christensen, 1972, variation II)

Let G be a group and $A \subseteq G$. The set A is called a generalized Haar null set if there exists a probability measure μ and a universally measurable set B such that $A \subseteq B$ and $\mu(gBh) = 0$ for every $g, h \in G$. Notation: $\mathcal{GHN}(G)$.

From now on, G will be a Polish group.

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Theorem

The system of (generalized) Haar null sets form a σ-ideal.

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If the group G is locally compact then a set $A \subseteq G$ has Haar measure zero $\iff$ A is Haar null $\iff$ A is generalized Haar null (so this is really a generalization of the notion of a Haar measure zero set).

If we drop the locally compactness assumption, what is the relation between variation I and II?

Theorem (Solecki)

Let G be a non-locally compact Abelian group. Then there exists a coanalytic set A such that $A \in \mathcal{GHN}(G) \setminus \mathcal{HN}(G)$.

Remark: in the non-locally compact case there exists continuum many pairwise disjoint closed sets and none of them is Haar null (in contrast to the locally compact case).

Relation of Haar null sets to Haar measure zero sets

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A subset A of G is Haar null/generalized Haar null $\iff$ there exists a Borel set/universally measurable set B and a continuous function $f : 2^\omega \longrightarrow G$ such that $A \subseteq B$ and $f^{-1}(gBh)$ has measure zero for every g, h.

This latter theorem allows us to introduce Haar meager and generalized Haar meager sets.

Definition (Darji, 2013)

A subset A of G is Haar meager if there exists a Borel set B and a continuous function $f : 2^\omega \longrightarrow G$ such that $A \subseteq B$ and $f^{-1}(gBh)$ is meager for every g, h.
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If G is locally compact, then $\mathcal{HM}(G) = \mathcal{M}(G)$.

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Category analogue to universal measurability?

What should be the definition of generalized Haar meager sets?

Definition

A subset B of a Polish space X is called universally Baire if for every continuous function $f : \omega^\omega \longrightarrow X$, the set $f^{-1}(B)$ has the property of Baire.

CAUTION: a set theorist may call such a set ω-universally Baire.

Theorem (Elekes, P.)

For a subset A of a Polish space X the followings are equivalent:
(1) the set A is universally Baire,
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$\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$. If G is locally compact, then $\mathcal{GHM}(G) = \mathcal{M}(G)$.

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Let G be a non-locally compact Abelian group, then:

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From now on, G denotes the Polish (non-locally compact) group $\mathbb{Z}^\omega$, but many results survive either if the group G has a two-sided invariant metric or if G admits a continuous surjective homomorphism onto a non-discrete locally compact Polish group H.

Theorem (Elekes, Poór, Vidnyánszky)

$$\mathrm{add}(\mathcal{HN}(G)) = \omega_1$$
$$\mathrm{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\},$$
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