Hyperfiniteness on topological Ramsey spaces

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Joint work with Zoltán Vidnyánszky

Borel equivalence relations

Definition

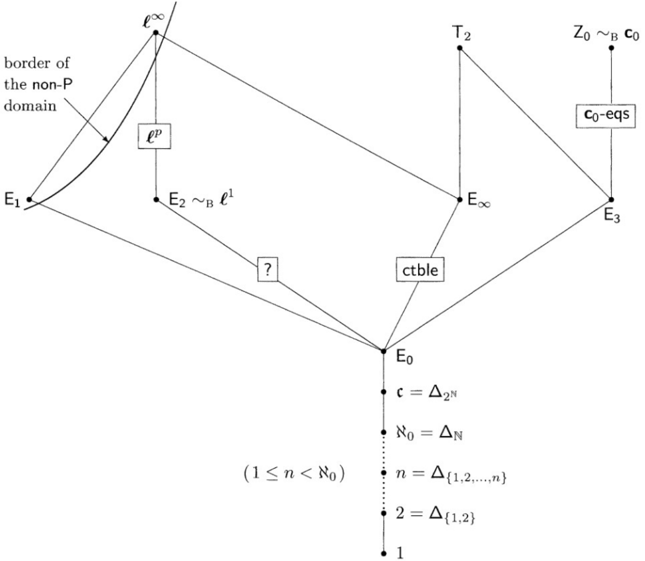
A *Borel equivalence relation* is an equivalence relation E on a standard Borel space X, which is Borel as a subset of $X \times X$.

Definition

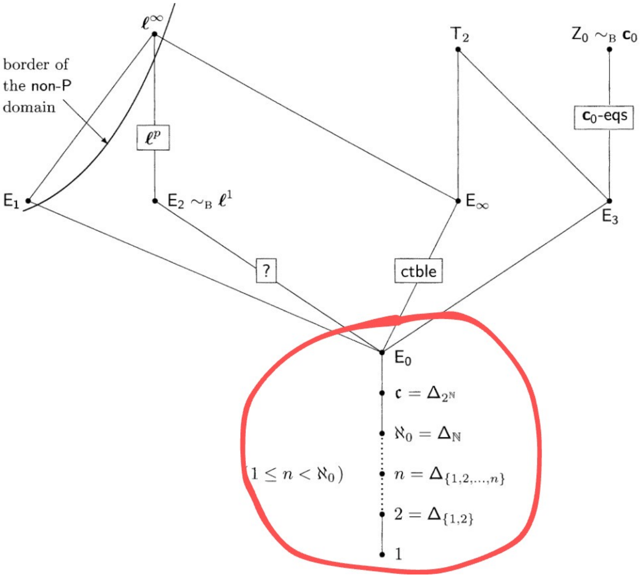
A Borel equivalence relation E on a standard Borel space X is *countable*, if all E-classes are countable.
Abbr.: CBER

Definition

Let X, Y be standard Borel spaces. An equivalence relation $E \subseteq X \times X$ is said to be *Borel reducible* to an equivalence relation $F \subseteq Y \times Y$, denoted $E \leq_B F$, if there exists a Borel map $\varphi : X \to Y$ such that $x_1 E x_2 \Leftrightarrow \varphi(x_1) F \varphi(x_2)$.



Reducibility between the key equivalence relations [3]



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Hyperfiniteness

Definition

A CBER E is *hyperfinite,* if there exist CBERs $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ with finite classes such that $E = \bigcup_{n \in \mathbb{N}} E_n$.

Example

For $x, y \in 2^{\mathbb{N}}$ let $x \mathbb{E}_0 y \Leftrightarrow (\exists n)\ (\forall k \geq n)\ x(k) = y(k)$.

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Theorem

The following are equivalent for a Borel equivalence relation E on a standard Borel space X:

- E *is hyperfinite*
- E *is induced by a Borel action of* $\mathbb{Z}$
- $E \leq_B \mathbb{E}_0$ *and* E *is countable*

Hyperfiniteness

Definition

A CBER E on X is *measure-hyperfinite* if for any Borel probability measure μ on X there exists a Borel set B with $\mu(B) = 1$ such that $E \restriction B$ is hyperfinite.

Problem

Is every measure-hyperfinite equivalence relation hyperfinite?

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Theorem (Hjorth-Kechris [2], Sulliwan-Weiss-Wright [8], Woodin)

Let E be a CBER on a standard Borel space. Then there is a comeager invariant Borel set C such that $E \restriction C$ is hyperfinite.

Topological Ramsey spaces - the Ellentuck space

- set of infinite objects: $[\mathbb{N}]^{\mathbb{N}}$ $A, B, C, \ldots$
- quasiorder: $A \leq B \Leftrightarrow A \subseteq B$
- finite approximations: $[\mathbb{N}]^{<\mathbb{N}}$ $a, b, c, \ldots$

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Theorem (Galvin-Prikry)

For any finite Borel coloring $c : [\mathbb{N}]^{\mathbb{N}} \to k$ there exists a set $A \in [\mathbb{N}]^{\mathbb{N}}$ such that c is constant on $[A]^{\mathbb{N}}$.

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Meta theorem 1. (Todorčević [9])

For any infinite object A and finite nice coloring of the infinite objects there exists $B \leq A$ such that $\{C : C \leq B\}$ is monochromatic.

Topological Ramsey spaces

Notation

$[b, A] = \{B : \ B \leq A \ \text{ and } B \ \text{ has approximation } b\}$

Meta theorem 2. (Todorčević [9])

Let A be an infinite object and let b be approximation. For every finite nice coloring $c : [b, A] \to k$ there exists $B \in [b, A]$ such that $[b, B]$ is monochromatic.

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Definition

A set $\mathcal{X}$ of infinite objects is *Ramsey* if for every infinite object A and finite approximation b there is some $B \in [b, A]$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$.
A set $\mathcal{N}$ of infinite objects is *Ramsey null* if for every infinite object A and finite approximation b there is some $B \in [b, A]$ such that $[b, B] \cap \mathcal{N} = \emptyset$.

Topological Ramsey spaces

Theorem (Mathias [5], Soare [7])

Let E be a CBER on $[\mathbb{N}]^{\mathbb{N}}$. Then E is hyperfinite on a Ramsey positive set, i.e. there is $A \in [\mathbb{N}]^{\mathbb{N}}$ such that $E \restriction [A]^{\mathbb{N}}$ is hyperfinite.

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Is the analogue true on other topological Ramsey spaces?

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Theorem (B.-Vidnyánszky [1])

Let $\mathcal{R}$ be a topological Ramsey space and E be a CBER on $\mathcal{R}$. Then there is a Ramsey positive set $A \subseteq \mathcal{R}$ such that $E \restriction A$ is hyperfinite.

Case of the Ellentuck space 1.

Suppose that E is a CBER on $[\mathbb{N}]^{\mathbb{N}}$. As all classes of E are countable, there are Borel involutions φ_n on $[\mathbb{N}]^{\mathbb{N}}$ such that $E = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(\varphi_n)$.
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Let $G_n = \bigcup_{i\in n} \mathrm{graph}(\varphi_i)$

It suffices to construct $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that

- The first n elements are the same for $\{A_k : k \geq n\}$
- $G_n \restriction [A_n]^{\mathbb{N}} \subseteq \mathbb{E}_0$ for every $n \in \mathbb{N}$

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Suppose A_n is given, let t be the set of n smallest elements of it. Let $A = A_n \setminus t$, the following lemma gives some $A' \subseteq A$, take $A_{n+1} = t \cup A'$.

Lemma

Let G be a bounded degree Borel graph on $[\mathbb{N}]^{\mathbb{N}}$, and suppose that $\max t < \min A$ for $t \in [\mathbb{N}]^n$ and $A \in [\mathbb{N}]^{\mathbb{N}}$. Then there exists $A' \subseteq A$ such that $G \upharpoonright [t \cup A']^{\mathbb{N}} \subseteq \mathbb{E}_0$.

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We define a bounded degree graph G^* on $[A]^{\mathbb{N}}$ by

$$(B, C) \in G^* \iff \big(B \neq C \wedge \exists r, s \subseteq t : (B \cup r, C \cup s) \in G\big).$$

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By a classical result of Kechris-Solecki-Todorčević, every bounded degree Borel graph admits a finite Borel vertex coloring. Let $c : [A]^{\mathbb{N}} \to k$ be such a coloring of G^*.

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By a classical result of Kechris-Solecki-Todorčević, every bounded degree Borel graph admits a finite Borel vertex coloring. Let $c : [A]^{\mathbb{N}} \to k$ be such a coloring of G^*.

By the Galvin-Prikry theorem, there exist $A' \subseteq A$ such that c is constant on $[A']^{\mathbb{N}}$.
But this means that $B, C \in [t \cup A']^{\mathbb{N}}$ cannot be G-related, unless $B \setminus t = C \setminus t$, yielding $B \mathbb{E}_0 C$. □

A combinatorial theorem

Definition

Let G be a graph, $B \subseteq V(G)$. The set B is called k-*separated* if for any $x \neq x' \in B$ we have $\mathrm{dist}_G(x, x') > k$.

Theorem (B.-Vidnyánszky [1])

Let E be a CBER on the space X, let $(G_n)_{n \in \mathbb{N}}$ be an increasing sequence of bounded degree Borel graphs such that $\bigcup_n G_n = E$, and let $f : \mathbb{N} \to \mathbb{N}$ be such that $\forall n \in \mathbb{N}\ f(n+1) \geq 2 \cdot (f(n)+1)$. Moreover, assume that $B_n \subseteq X$ are Borel sets so that every B_n is $f(n)$-separated in G_n.
Then $E \restriction B$ is hyperfinite, where $B = \{x : \exists^\infty n\ (x \in B_n)\}$.

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Then $E \restriction B$ is hyperfinite, where $B = \{x : \exists^\infty n\ (x \in B_n)\}$.

By this theorem it suffices to find $f(n)$-separated sets B_n such that $\{x : \exists^\infty n\ (x \in B_n)\}$ is Ramsey positive

Questions

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Let E be a CBER on $[\mathbb{N}]^{\mathbb{N}}$. Is there a Ramsey co-null Borel set B such that $E \restriction B$ is hyperfinite?

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Question (Wang-Panagiotopoulos [6])

Characterize those topological Ramsey spaces on which any CBER is smooth on some positive set.

Thank you!

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