Generalized Tukey relations in Solovay model

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Preliminary

In this talk, we work in ZFC unless otherwise stated.

(Some theorems are stated in ZF or ZF + DC, but each time they are specified)

We consider the following form of the Solovay model.

Definition (Solovay model)

Let κ be an inaccessible cardinal and G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter over V. Then we call $V(\mathbb{R}^{V[G]}) = \mathrm{HOD}^{V[G]}_{V \cup \mathbb{R}^{V[G]}}$ the **Solovay model** for κ.

Uncountable liner order

There are previous researches about combinatorial objects in specific choice-less models, such as $L(\mathbb{R})$ satisfying AD and the Solovay model.
Especially, there are results about uncountable linear orders.

Theorem (Chan-Jackson)

If $L(\mathbb{R}) \models$ AD then $L(\mathbb{R}) \models$ " there is no Suslin line ".

Theorem (Woodin)

Let $V(\mathbb{R}^{V[G]})$ be the Solovay model for a weakly compact cardinal.
Then $V(\mathbb{R}^{V[G]}) \models$ " there is no Suslin line ".

Uncountable liner order

Theorem (Sakai-T.)

- If $L(\mathbb{R}) \models \mathrm{AD}$ then $L(\mathbb{R}) \models$ " there is no Aronszajn line ".
- Let $V(\mathbb{R}^{V[G]})$ be the Solovay model for a weakly compact cardinal. Then $V(\mathbb{R}^{V[G]}) \models$ " there is no Aronszajn line ".

That is, in such models any uncountable linear order contains a subset which is order isomorphic to $(\omega_1, <), (\omega_1, >)$ or an uncountable subset of the real line $\mathbb{R}$ (three element basis!).

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$\rightsquigarrow$ How about other classes of order sets?

In particular, how about directed sets in the Solovay model?

Directed set

In this talk, we compare directed sets associated with cardinal invariants.

Example

$\mathfrak{d}$ is the least cardinality of a cofinal subset of $(\omega^\omega, \leqslant^*)$, where

$$x \leqslant^* y \Leftrightarrow x(n) \leqslant y(n) \text{ for all but finitely many } n \in \omega.$$

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In choice-less models, it is impossible to compare cardinalities of cofinal subsets, however we can compare cofinal types through the existence of certain maps.

Tukey relation in ZFC

In ZFC, the Tukey relation plays important role in investigating directed sets.

Definition (Tukey relation)

Let D and E be directed sets.

- $f : D \to E$ is **Tukey map** if f maps unbounded sets in D into unbounded sets in E.
- $D \leq_T E$ (D is **Tukey reducible** to E) if there is a Tukey map $D \to E$.
- $D \equiv_T E$ (D is **Tukey equivalent** to E) if $D \leq_T E$ and $E \leq_T D$.

$\equiv_T$ is an equivalent relation between directed sets.

An equivalent class of $\equiv_T$ is called a **cofinal type**.

Tukey relation in ZFC

The following are important consequences of the Tukey reducibility.

Fact (ZFC)

Let D and E be directed sets. $D \leq_T E$ if and only if there is a map from E to D which maps cofinal subsets in E into cofinal subsets in D. In particular, if $D \leq_T E$ then the cofinality of D is less than or equal to that of E.

Fact (ZFC)

Let D and E be directed sets. $D \equiv_T E$ if and only if there is a directed set in which both D and E can be embedded as cofinal subsets.

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Fact (ZFC)

Let D and E be directed sets. $D \equiv_T E$ if and only if there is a directed set in which both D and E can be embedded as cofinal subsets.

The proofs of these theorems require AC.

$\rightsquigarrow$ We introduce a generalization of Tukey reducibility that works well without AC.

Pre-Tukey relation

Definition

For an ordered set P, let P^* be the set of all non-empty upward closed subsets of P, which is ordered by reverse inclusion.

Note that P^* is the maximum completion of P w.r.t. the Tukey relation.

Definition (pre-Tukey relation)

Let D and E be directed sets.

- $D \leq_T^* E$ (D is **pre-Tukey reducible** to E) if $D \leq_T E^*$
- $D \equiv_T^* E$ (D is **pre-Tukey equivalent** to E) if $D \leq_T^* E$ and $E \leq_T^* D$.

$\leq_T^*$ is a pre-ordering and $\equiv_T^*$ is an equivalent relation.

Pre-Tukey map

The pre-Tukey relation is characterized by maps that generalize Tukey maps.

Lemma (ZF)

The following are equivalent for directed sets D and E.

- $D \preceq_T^* E$.
- there is a function $\pi \colon D \to \mathcal{P}(E)$ such that $\pi(d) \neq \varnothing$ for any $d \in D$, and

$$\forall e \in E \; \exists d \in D \; \forall d' \in D \; (e \in \pi(d') \Rightarrow d' \leqslant_D d).$$

We call maps $\pi \colon D \to \mathcal{P}(E)$ satisfying above property **pre-Tukey maps** from D to E.

Tukey and Pre-Tukey map

The existence of Tukey maps is equivalent to the existence of pre-Tukey maps in ZFC.

Lemma (ZF)

Let D and E be directed sets.

- For $f\colon D \to E$ define $\pi_f\colon D \to \mathcal{P}(E)$ by $\pi(d) = \{e \in E \mid f(d) \leqslant_E e\}$.
  If f is a Tukey map then π_f is a pre-Tukey map.
- For $\pi\colon D \to \mathcal{P}(E)$ suppose that $f\colon D \to E$ satisfies $f(d) \in \pi(d)$ for all $d \in D$.
  If π is a pre-Tukey map then f is a Tukey map.

Pre-Tukey map

The pre-Tukey relation on directed sets works well without AC.

Theorem (ZF)

Let D and E be directed sets. $D \equiv^*_T E$ if and only if there is a directed set in which both D and E can be embedded as cofinal subsets.

We consider cofinal types of directed sets (that is, equivalent classes w.r.t. $\equiv^*_T$), especially in the Solovay model.

Directed sets associated with cardinal invariants

We consider directed sets $(\omega^\omega, \leqslant^*), (\mathcal{M}, \subseteq)$ and $(\mathcal{N}, \subseteq)$, where $\mathcal{M}$ is the ideal of meager sets and $\mathcal{N}$ is the ideal of null sets in 2^ω.
In ZFC, it holds that $\mathfrak{d} \leqslant \mathrm{cof}(\mathcal{M}) \leqslant \mathrm{cof}(\mathcal{N})$, the corresponding cardinal invariants with these directed sets.

$$
\begin{array}{ccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \rightarrow & \mathrm{non}(\mathcal{M}) & \rightarrow & \mathrm{cof}(\mathcal{M}) & \rightarrow & \mathrm{cof}(\mathcal{N}) & \rightarrow & 2^{\aleph_0} \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & \uparrow & & \mathfrak{b} & \rightarrow & \mathfrak{d} & & \uparrow & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \rightarrow & \mathrm{add}(\mathcal{N}) & \rightarrow & \mathrm{add}(\mathcal{M}) & \rightarrow & \mathrm{cov}(\mathcal{M}) & \rightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}
$$

Directed sets associated with cardinal invariants

Fact (ZFC)

$(\omega^\omega, \leqslant^*) \preceq_T (\mathcal{M}, \subseteq)$ *and* $(\mathcal{M}, \subseteq) \preceq_T (\mathcal{N}, \subseteq)$.

The construction of these Tukey maps uses AC, as it requires choosing a countable family of closed (open) sets for each F_σ (G_δ) set.

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The construction of these Tukey maps uses AC, as it requires choosing a countable family of closed (open) sets for each F_σ (G_δ) set.
However, these construction can be adapted to construct pre-Tukey maps.

Lemma (ZF + DC)

$(\omega^\omega, \leqslant^*) \preceq_T^* (\mathcal{M}, \subseteq)$ and $(\mathcal{M}, \subseteq) \preceq_T^* (\mathcal{N}, \subseteq)$.

So we consider the pre-Tukey relations in the reverse directions.

Main results : distinct cofinal types

We prove that there are no pre-Tukey maps between these directed sets.

Theorem (Sakai-T.)

Let $V(\mathbb{R}^{V[G]})$ be the Solovay model for an inaccessible cardinal. In $V(\mathbb{R}^{V[G]})$,

- $(\mathcal{M}, \subseteq) \not\leqslant_T^* (\omega^\omega, \leqslant^*)$.
- $(\mathcal{N}, \subseteq) \not\leqslant_T^* (\mathcal{M}, \subseteq)$.

In the sense of the pre-Tukey relation, their cofinal types are distinct in the Solovay model.

Sketch of proof (1/3)

We see the sketch of the proof of $(\mathcal{M}, \subseteq) \not\leqslant^*_T (\omega^\omega, \leqslant^*)$

Let $V(\mathbb{R}^{V[G]})$ be the Solovay model for an inaccessible cardinal.

We show that there is no pre-Tukey map from $(\mathcal{M}, \subseteq)$ to $(\omega^\omega, \leqslant^*)$ in $V(\mathbb{R}^{V[G]})$.

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Lemma

Assume that there is a pre-Tukey map in $V(\mathbb{R}^{V[G]})$ from $(\mathcal{M}, \subseteq)$ to $(\omega^\omega, \leqslant^*)$ which is definable in $V[G]$ with parameters from V. Then for all $x \in \omega^\omega$ there is $y \in \omega^\omega$ such that for all $z \in \omega^\omega$ $(y \leqslant^* z \Rightarrow C_z \subseteq C_x)$, where C_x is the collection of Cohen reals over $V[x]$.

By this Lemma, we can reduce the absence of pre-Tukey maps to the existence of some generic reals.

Sketch of proof (2/3)

We work in $V[G]$.
By Lemma, it suffices to find $x \in \omega^\omega$ such that for all $y \in \omega^\omega$ there are $z \in \omega^\omega$ and $w \in \omega^\omega$ such that $y \leqslant^* z$ and w is Cohen over $V[z]$ and not Cohen over $V[x]$.

We can see that Laver reals are cofinal in $(\omega^\omega, \leqslant^*)$ in $V[G]$.

Lemma

Let $\mathbb{L}$ be the Laver forcing notion. In $V[G]$ for all $y \in \omega^\omega$ and $S \in \mathbb{L}$ there is $T \leqslant_{\mathbb{L}} S$ such that $T \Vdash_{\mathbb{L}}$ " $y \leqslant^* \dot{z}$ ", where $\dot{z}$ is a canonical name of $\mathbb{L}$-generic real.

Sketch of proof (3/3)

Let $\lambda = (2^{2^\omega})^V$ and $x \in \omega^\omega$ be a real coding a surjection $\omega \to \lambda$. Suppose $y \in \omega^\omega$. We want $z \in \omega^\omega$ and $w \in \omega^\omega$ such that $y \leqslant^* z$ and w is Cohen over $V[z]$ and not Cohen over $V[x]$.

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We take $S \in \mathbb{L}^{V[x]}$ which is $(V, \mathbb{L})$-generic. Then we can take $\mathbb{L}^{V[x]}$-generic filter I over $V[x]$ with $S \in I$ and $y \leqslant^* \bigcup \bigcap I =: z$.

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Let

$$\Gamma = \{c \in V[z] \mid c \text{ is a Borel code of a meager set}\}.$$

Since Γ is countable in $V[x][z]$, $A = \bigcup_{c \in \Gamma} B_c^{V[x][z]}$ is meager in $V[x][z]$.
So we can take $w \in (\omega^\omega)^{V[x][z]} \setminus A$. Then w is Cohen over $V[z]$ and not Cohen over $V[x]$ (Laver forcing does not add Cohen reals). □

$(\omega_1, \leqslant)$ and $([\omega^\omega]^\omega, \subseteq)$

In ZFC, $(\omega_1, \leqslant)$ and $([\omega^\omega]^\omega, \subseteq)$ has cofinality ω_1 and $2^{\aleph_0}$ respectively.

For these directed sets, we show the following.

Theorem (Sakai-T.)

In the Solovay model $V(\mathbb{R}^{V[G]})$ for an inaccessible cardinal,

- $(\omega^\omega, \leqslant^*) \not\leq_T^* (\omega_1, \leqslant)$ and $(\omega_1, \leqslant) \not\leq_T^* (\mathcal{N}, \subseteq)$.
- $(\omega_1, \leqslant) \leq_T^* ([\omega^\omega]^\omega, \subseteq)$ and $(\mathcal{N}, \subseteq) \leq_T^* ([\omega^\omega]^\omega, \subseteq)$
- $([\omega^\omega]^\omega, \subseteq) \not\leq_T^* (\mathcal{N}, \subseteq)$.

Consequences

In the following figure, $D \to E$ means that $D \preceq_T^* E$, and $D \dashrightarrow E$ means that $D \npreceq_T^* E$ for directed sets D and E. In particular, in the Solovay model, the cofinal types of these directed sets are different each other.

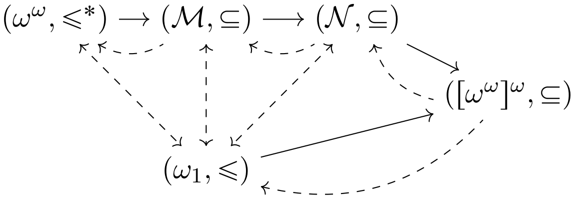

Figure: pre-Tukey relations in the Solovay model

Questions (1/2) : other directed sets

In set theory, there are various directed sets induced by ideals or filters.

Question 1

- In the Solovay model, do directed sets induced by different structures (especially those related to the real) have distinct cofinal types?
- Is there a non-trivial pair of directed sets which are pre-Tukey equivalent?

Questions (2/2) : determinacy

As mentioned about uncountable linear orders in the first section, it is known that some similar combinatorial propositions hold in both the Solovay model and $L(\mathbb{R})$ satisfying AD.

Question 2

Assume ZF $+ V = L(\mathbb{R}) +$ AD. Do the same separations hold as in the Solovay model?