Good colorings for stationary lists

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1 Background

2 Flow of Komjáth's 2nd theorem

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Chromatic number

Let $X = (V_X, E_X)$ be an (undirected simple) graph.

Definition

A function f whose domain is V_X is called *a coloring of* X. Also, if $f(v) \neq f(w)$ holds for every $\{v, w\} \in E_X$, then f is termed *a good coloring*.

Definition

We define *the chromatic number of* X as

$$\mathrm{Chr}(X) = \min\{\kappa \in \mathsf{Card} \mid \exists f : V_X \to \kappa[f \text{ is good.}]\}.$$

Let X be a bipartite graph, that is, $X = (A \sqcup B, E_X)$ such that $\{v, w\} \in E_X \to v \in A \wedge w \in B$. Then, $\mathrm{Chr}(X) \leq 2$, since the following function is a good coloring:

$$f(v) = \begin{cases} 0 \text{ (if } v \in A) \\ 1 \text{ (if } v \in B) \end{cases}$$

List chromatic number

Definition

We define *the list chromatic number* $\mathrm{List}(X)$ as

$$\min\left\{\kappa \in \mathrm{Card} \mid \forall L : V_X \to [\mathrm{On}]^{\kappa} \exists f : V_X \to \mathrm{On}[\forall v \in V_X(f(v) \in L(v)) \wedge f \text{ is good.}]\right\}.$$

Definition

We define *the restricted list chromatic number* $\mathrm{List}^*(X)$ as

$$\min\left\{\kappa \in \mathrm{Card} \mid \forall L : V_X \to [\kappa]^{\kappa} \exists f : V_X \to \kappa[\forall v \in V_X(f(v) \in L(v)) \wedge f \text{ is good.}]\right\}.$$

Coloring number

Definition

The coloring number $\mathrm{Col}(X)$ is the least $\kappa \in \mathrm{Card}$ such that

$$\exists \text{ well order } \prec \text{ on } V_X \forall v \in V_X [|\{w \in V_X \mid w \prec v\}| < \kappa].$$

$\mathrm{Chr}(X) \leq \mathrm{List}^*(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X)$

Proposition

$\mathrm{Chr}(X) \leq \mathrm{List}^*(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X)$

Proof.

($\mathrm{Chr}(X) \leq \mathrm{List}^*(X)$) Suppose that $\mathrm{List}^*(X) \leq \kappa$ and let $L : V_X \to \{\kappa\}$ be a list function. We show that $\mathrm{List}^*(X) \leq \kappa$. If there exists a good coloring $f : V_X \to \kappa$ such that $f(v) \in L(v) = \kappa$, then this f is a witness of $\mathrm{Chr}(X) \leq \kappa$.
($\mathrm{List}^*(X) \leq \mathrm{List}(X)$) This is clear, since $[\kappa]^\kappa \subseteq [\mathrm{On}]^\kappa$.
($\mathrm{List}(X) \leq \mathrm{Col}(X)$) Suppose that $\mathrm{Col}(X) \leq \kappa$. We show that $\mathrm{List}(X) \leq \kappa$. Let $\prec$ be a well-ordering of V_X that is a witness of $\mathrm{Col}(X) \leq \kappa$. Take a list function $L : V_X \to [\mathrm{On}]^\kappa$. We construct a good coloring f with $f(v) \in L(v)$ by induction on $(V_X, \prec)$. Let $v \in V_X$ and suppose that for all $w \prec v$, $f(w)$ is already defined. By the choice of $\prec$, $S = |\{f(w) \mid w \prec v\}| < \kappa$. Then we can choose $f(v) \in L(v) \setminus S \neq \emptyset$. This coloring f is good by the construction. $\square$

Komjáth's results

In [1], Komjáth investigated these characteristics. In this talk, we focus on the following two results:

Theorem(Komjáth)

It is consistent that there exists a graph X such that $\text{List}(X) \neq \text{List}^*(X)$.

Theorem(Komjáth)

Under GCH, $\text{Col}(X) \leq \text{List}(X)^+$ and $\text{Col}(X) \leq \text{List}^*(X)^{++}$ hold.
(Recall:$\text{List}^*(X) \leq \text{List}(X) \leq \text{Col}(X)$)

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Flow chart

Theorem(Komjáth, repeat)

Under GCH, $\mathrm{Col}(X) \leq \mathrm{List}(X)^{+}$ and $\mathrm{Col}(X) \leq \mathrm{List}^{*}(X)^{++}$ hold.

Komjáth's 2nd theorem is proved by contraposition. For example, the first one is showed as follows.

Suppose that $\mathrm{Col}(X) > \mu^{+}$ and X is "minimal".

↓

Find a bipartite subgraph Y

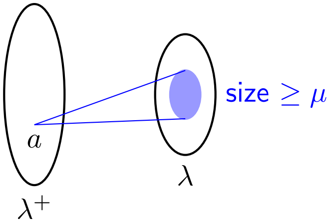
↓

Obtain $\mathrm{List}(Y) > \mu$

Find a bipartite subgraph

We may assume that for every subgraph $X' \subseteq X$ with $|X'| < |X|$, $\mathrm{Col}(X') \leq \mu^+$ (minimality). Then we can find a subgraph $Y = (A \sqcup B, E_Y)$ satisfying the following for some λ:

- $|A| = \lambda^+$,
- $|B| = \lambda$, and
- $\forall a \in A, |\{b \in B \mid \{a,b\} \in E_Y\}| \geq \mu$.

$Y =$ 

Suppose that $\mathrm{Col}(X) > \mu^+$ and X is "minimal".

↓

Find a bipartite subgraph Y

↓

Obtain $\mathrm{List}(Y) > \mu$

GCH result for bipartite graphs

Now, we focus on the subgraph $Y \subseteq X$ with

- $|A| = \lambda^+$,
- $|B| = \lambda$, and
- $\forall a \in A, |\{b \in B \mid \{a, b\} \in E_Y\}| \geq \mu$.

In [1], Komjáth proved the following:

Lemma(Komjáth)

Under GCH, every bipartite graph satisfying the above three conditions has a list chromatic number at least μ^+.

Therefore, $\mathrm{List}(Y) > \mu$, and of course, $\mathrm{List}(X) > \mu$.

Suppose that $\mathrm{Col}(X) > \mu^+$ and X is "minimal".

↓

Find a bipartite subgraph Y

↓

Obtain $\mathrm{List}(Y) > \mu$

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Notations

We introduce new notations for list colorings.

Definition

For a class $\mathcal{C}$, List$(X, \mathcal{C})$ is the following condition:

$$\forall L : V_X \to \mathcal{C} \exists f : V_X \to \bigcup \mathcal{C} [\forall v \in V_X (f(v) \in L(v)) \land f \text{ is good.}]$$

For example, List$(X, [\mathrm{On}]^\kappa)$ (resp. List$(X, [\kappa]^\kappa)$) deals with the condition of List(X) (resp. List$^*(X)$).
We are interested in the condition List$(X, \mathrm{Stat}_\kappa)$ for regular κ. Since $\mathrm{Stat}_\kappa \subseteq [\kappa]^\kappa$, List$(X, [\kappa]^\kappa) \to$ List$(X, \mathrm{Stat}_\kappa)$ holds.

Separation

Theorem(Sakai, H, repeat)

For every regular κ, there exists a graph X such that List(X, Stat_κ) holds, but List$(X, [\kappa]^\kappa)$ fails.

Proof.

We show that the complete bipartite graph $X = (A \sqcup B, E_X)$ with $|A| = 2^\kappa$ and $|B| = \kappa$ is as desired.
(List(X, Stat_κ)) Take $L : A \sqcup B \to \text{Stat}_\kappa$. We can find a good coloring function f of L as follows. We may assume that $B = \kappa$. First, for $\gamma \in B$, choose $f(\gamma) \in L(\gamma) \setminus \gamma$. Then, $N = \{f(\gamma) \mid \gamma \in B\}$ is nonstationary. Thus, we can choose $f(a) \in L(a) \setminus N$ for $a \in A$. Clearly, this f is as desired. □

Separation

Theorem(Sakai, H)

For every regular κ, there exists a graph X such that $\mathsf{List}(X, \mathsf{Stat}_\kappa)$ holds, but $\mathsf{List}(X, [\kappa]^\kappa)$ fails.

Proof.

$(\neg\mathsf{List}(X, [\kappa]^\kappa))$ Identify A and B with $[\kappa]^\kappa$ and κ, respectively. Define a list function $L : A \sqcup B \to [\kappa]^\kappa$ by

$$\begin{cases} L(X) = X(\text{if } X \in A) \\ L(\gamma) = \kappa \setminus \gamma(\text{if} \gamma \in B) \end{cases}.$$

Let f be a choice function of L and show that f is not good. By the definition of L, $f[B] \in [\kappa]^\kappa = A$. Therefore $f(f[B]) \in L(f[B]) = f[B]$ and we can take a $\gamma \in B$ with $f(\gamma) = f(f[B])$. Since X is the complete bipartite graph, $\{\gamma, f[B]\} \in E_X$, so f is not good. □

Revised Komjáth's 2nd theorem

In the proof of $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$, the following is a key lemma.

Lemma(Komjáth, repeat)

Under GCH, every bipartite graph satisfying the three conditions has a list chromatic number of at least μ^+.

Under our notation, this can be rewritten as follows:

Lemma(Komjáth)

Under GCH, for every bipartite graph X satisfying the three conditions, $\mathrm{List}(X, [\mathrm{On}]^\mu)$ fails.

Suppose that $\mathrm{Col}(X) > \mu^+$ and X is "minimal".

↓

Find a bipartite subgraph Y

↓

Obtain $\mathrm{List}(Y) > \mu$

Revised Komjáth's 2nd theorem

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Under GCH, every bipartite graph satisfying the three conditions has a list chromatic number of at least μ^+.

Under our notation, this can be rewritten as follows:

Lemma(Komjáth)

Under GCH, for every bipartite graph X satisfying the three conditions, $\mathrm{List}(X, [\mathrm{On}]^\mu)$ fails.

Suppose that $\mathrm{Col}(X) > \mu^+$ and X is "minimal".

↓

Find a bipartite subgraph Y

↓

Obtain $\neg\mathrm{List}(Y, [\mathrm{On}]^\mu)$

Our 2nd result

Theorem(Komjáth)

Under GCH, if $\mathrm{Col}(X) > \mu^+$ then $\mathrm{List}(X, [\mathrm{On}]^\mu)$ fails for every μ.

Theorem(Sakai, H)

Under GCH, if $\mathrm{Col}(X) > \mu^{++}$ then $\mathrm{List}(X, \mathrm{Stat}_\mu)$ fails for every regular μ.

Key lemma

Theorem(Sakai, H, repeat)

Under GCH, if Col$(X) > \mu^{++}$ then List(X, Stat_μ) fails for every regular μ.

Suppose that Col$(X) > \mu^{++}$. By the same argument of Komjáth's proof, we can find a bipartite subgraph $Y = (A \sqcup B, E_Y)$ satisfying

- $|A| = \lambda^+$,
- $|B| = \lambda$, and
- $\forall a \in A, |\{b \in B \mid \{a, b\} \in E_Y\}| \geq \mu^+$.

for some λ. We proved the following:

Lemma

Under GCH, for every bipartite graph X and every regular cardinal μ satisfying the above three conditions, List(X, Stat_μ) fails.

Questions

Theorem(Sakai, H, repeat)

Under GCH, if $\mathrm{Col}(X) > \mu^{++}$ then $\mathrm{List}(X, \mathrm{Stat}_\mu)$ fails for every regular μ.

Question

Is the above result optimal? In other words, is it consistent with GCH that there is a graph X and a regular cardinal μ such that $\mathrm{Col}(X) = \mu^{++}$ and $\mathrm{List}(X, \mathrm{Stat}_\mu)$ holds?

Question

What about other classes? For example, for a filter on μ, for which assumptions does $\mathrm{List}(X, F)$ fail in general?

Reference I

[1] Péter Komjáth.
The list-chromatic number of infinite graphs.
Israel Journal of Mathematics, 196(1):67–94, 2013.

Key lemma

Theorem(Sakai, H, repeat)

Under GCH, if $\mathrm{Col}(X) > \mu^{++}$ then $\mathrm{List}(X, \mathrm{Stat}_\mu)$ fails for every regular μ.

Suppose that $\mathrm{Col}(X) > \mu^{++}$. By he same argument of Komjáth's proof, we can find a bipartite subgraph $Y = (A \sqcup B, E_Y)$ satisfying

- $|A| = \lambda^+$,
- $|B| = \lambda$, and
- $\forall a \in A, |\{b \in B \mid \{a, b\} \in E_Y\}| \geq \mu^+$.

for some λ. Thus, it is sufficient to show that:

Lemma

Under GCH, for every bipartite graph X and every regular cardinal μ satisfying the above three conditions, $\mathrm{List}(X, \mathrm{Stat}_\mu)$ fails.

Good coloring of bipartite graphs for stationary lists

Lemma

Under GCH, for every bipartite graph $X = (A \sqcup B, E_X)$ and every regular cardinal μ satisfying

- $|A| = \lambda^+$,
- $|B| = \lambda$, and
- $\forall a \in A, |\{b \in B \mid \{a, b\} \in E_X\}| \geq \mu^+$,

$\mathrm{List}(X, \mathrm{Stat}_\mu)$ fails.

Sketch of the proof

Let X be a such graph. We construct a list $L : A \sqcup B \to \mathrm{Stat}_\mu$ which has no good coloring function. We may assume that $B = \lambda$. By taking a subgraph of X, if necessary, we may assume that this λ is minimal with respect to the above three conditions. That is, we can assume that if there are $A' \subseteq A$, $B' \subseteq B$, and ν such that $|A| = \nu^+$, $|B| = \nu$, and $\forall a \in A', |\{b \in B' \mid \{a, b\} \in E_X\}| \geq \mu^+$, then $\nu = \lambda$.

Good coloring of bipartite graphs for stationary lists

Sketch of the proof

Thus, by the minimality of λ, $B_a = \{b \in B \mid \{a, b\} \in E_X\}$ is unbounded in λ for all $a \in A$. Also, by removing edges, we assume that B_a has size μ^+. In other words, we may assume that $\mathrm{cf}(\lambda) = \mu^+$ and B_a is cofinal in λ. Take a cofinal sequence $\langle \lambda_i \mid i < \mu^+ \rangle$ with $\lambda_0 = 0$. Since GCH holds, there is a $\subseteq^*$-sequence $\langle C_i \mid i < \mu^+ \rangle$ of clubs in μ such that

$$\forall \text{club } C \subseteq \mu \forall^\infty i < \mu^+ [C_i \subseteq^* C]$$

where $S \subseteq^* S'$ means $S \setminus S'$ is bounded in μ. For $\xi \in [\lambda_i, \lambda_{i+1}) \subseteq \lambda = B$, we define $F(\xi) = C_i$. By a diagonal argument, for every $a \in A$, we can find $\gamma_a < \mu$ such that

$$\forall \text{club } C \subseteq \mu \exists \xi \in B_a [F(\xi) \setminus (\gamma_a + 1) \subseteq C].$$

Good coloring of bipartite graphs for stationary lists

Sketch of the proof.

By the pigeonhole principle, there exists a $\gamma < \mu$ such that $A' = \{a \in A \mid \gamma_a = \gamma\}$ has size λ^+. We shrink A to A' and define the list function L as follows. For $\xi \in B$, we define $L(\xi) = F(\xi) \setminus \gamma$. Since GCH holds, we can identify A' with the set $\prod_{\xi \in B} L(\xi)$. Here, for $g \in A'$, define $L(g) = \{g(\xi) \mid \xi \in B_g\}$. This L has no good coloring function by a simple observation, so it only remains to check that $L(g)$ is stationary.

Take a club set C in μ. By the choice of γ and A', there is a $\xi \in B_g$ such that $B_g = F(\xi) \setminus (\gamma + 1) \subseteq C$. Therefore $g(^{\xi}) \in L(g) \cap C \neq \emptyset$, so we are done. □