On some applications of Δ-spaces and Δ_1-spaces

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Set Theory and Topology

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7. (MA) $\wedge(\sim$ (CH)): $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a Q-set (Martin-Solovay (1970), M.E. Rudin (1977)).

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8. Every pseudocompact Δ_1-space with countable tightness is scattered (J.K.-Kurka-Leiderman).

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2. A Tychonoff space X is a Δ-space iff $C_p(X)$ is distinguished, i. e., the strong dual of $C_p(X)$ is barrelled iff the strong dual of $C_p(X)$ carries the finest locally convex topology (J.K-Leiderman).

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3. This provides (with several applications) a significant connection between attractive problems from set theory related with Δ-sets, λ-sets, and Q-sets X, and their corresponding distinguished spaces $C_p(X)$.

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5. $C_k(X)$ has a fundamental sequence of bounded sets iff X is Warner bounded (Warner).

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5. *$C_k(X)$ does not contain a copy of ℓ_1.*
6. *Every continuous convex function with domain a nonempty open convex subset is Fréchet differentiable on a dense G_δ subset of its domain.*

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Let X be a compact space. The following assertions are equivalent: (i) The space X is a Δ_1-space. (ii) The space X is scattered. (iii) The space $C_k(X)$ is an Asplund space.

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Problem 5

Let X be a compact space. Find a "nice" property $\mathcal{P}$ on $C(X)$ or $C(X)'$ under which the following statement holds true. X is a Δ-space iff $C(X)$ is Asplund and $C(X)$ (or $C(X)'$) satisfies property $\mathcal{P}$.

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Any pseudocompact Δ-space of countable tightness must be scattered.

On ω-resolvable Δ-spaces and Δ_1-spaces

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3. One of the problems considered by Malykhin refers to the existence of irresolvable spaces (i.e. crowded not resolvable) satisfying the Baire Category Theorem. Under L=V (i.e. every set is constructible) there is no Baire irresolvable space (Kunen-Szymański-Tall).

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4. (V=L) $\Rightarrow$ (AC) $\wedge$ (GCH).

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3. Next fact was obtained as a step in the proof that a Δ-space pseudocompact of countable tightness is scattered.

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2. Consider some cases for which this problem has a positive answer.

1. Under Souslin hypothesis (SH) (i.e. there are no Souslin lines) if X is crowded and Baire with $c(X) = \omega$, then X is ω-resolvable (Casarrubias-Segura, Hernandez-Hernandez, Tamaris-Mascar) (Recall that $(MA) \wedge (\sim (CH)) \Rightarrow (SH)$.)

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(SH) If X is a Baire space with $c(X) = \omega$ and X is a Δ-space, then X has isolated points.

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Corollary 14

(SH) If X is a Baire space with $c(X) = \omega$ and X is a Δ-space, then X has isolated points.

2. The axiom of constructibility, $V = L$, implies that every Baire space without isolated points is ω-resolvable (Pavlov). Hence (under $V = L$) every Δ-space which is Baire has isolated points.

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Corollary 15

(MA) If X is a Baire space which is a Δ-space and that satisfies one of the mentioned above properties $\mathcal{P}$, then X has isolated points.

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Example 16

There exist crowded countably compact Δ_1-spaces not Δ-spaces which are ω-resolvable.

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Does every metrizable Baire with the Δ_1-property have an isolated point?

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3. Dealing with Δ_1-spaces we propose the following extension of Proposition 10 as follows.

Theorem 18 (J.K.-Leiderman-Tkachuk)

If X is a separable crowded Baire space, then X is not a Δ_1-space. Hence, a separable Baire space which is a Δ_1-space has isolated points.

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2. Indeed, Juhasz and van Mill showed that there exists a dense countably compact subspace $Z \subset \beta\omega \setminus \omega$ in which all countable subsets are scattered and hence Z is a Δ_1-space.
3. If $X = \omega \cup Z$, then X is a countably compact separable Δ_1-space not scattered. (J.K.-Leiderman-Tkachuk)

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Theorem 19 (J.K.-Leiderman-Tkachuk)

A space X is a Δ_1-space iff, for any function $f \in \mathbb{R}^X$ such that $f^{-1}(\mathbb{R} \setminus \{0\})$ is countable, there exists a bounded set $B \subset C_p(X)$ such that $f \in \overline{B}$.

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Theorem 20 (J.K.-Leiderman-Tkachuk)

If X has countable pseudocharacter, then X is a Δ_1-space iff for any function $f \in \mathbb{R}^X$ such that $f^{-1}(\mathbb{R} \setminus \{0\})$ is countable, there exists a sequence $\{f_n : n \in \omega\} \subset C_p(X)$ that converges to f.

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Corollary 21

All elements of the $\sum$-product $\Sigma(\mathbb{R}^X)$ are functions of the first Baire class iff X is a Δ_1-space.