A Small, Unruly Radon-Nikodym Compact Space

from a $\diamondsuit$

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Joint work with Arturo Martinez-Celis

Radon-Nikodym compacts

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Theorem (Orihuela, Schachermayer, Valdivia 1991)

Compact Hausdorff space is Eberlein iff it is RN and Corson.

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Theorem (Avilés 2005)

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Weight of a space X (or $w(X)$) is the smallest size of a basis of X.

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There is an RN-compact space with a continuous image which is not RN-compact. It has weight $\mathfrak{c}$.

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Theorem

There is such RN-compact space of weight ω_1 in many models with $\mathrm{non}(\mathcal{M}) = \omega_1$.
More precisely, under $\diamondsuit(\mathrm{non}(\mathcal{M}))$ which is true e.g. in Cohen, Sacks and Miller models where $\mathfrak{c} = \omega_2$.

Construction basis

Fix infinite $A = \dot{\bigcup}_{t \in 2^{<\omega}} A_t$ and B and $\langle C_b : b \in B \rangle$ with

- each C_b is a countable subset of A,
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Furthermore there is a function $D : B \to \omega^{2^{<\omega}}$ st

$$(\star) \qquad \begin{array}{c} \text{given } s : A \to \omega \text{ there is } b \in B \text{ for which} \\ \{a \in C_b \cap A_t : s(a) = D(b)(t)\} \text{ is infinite} \\ \text{for each } t \end{array}$$

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- A is discrete,
- the basis of neighbourhoods of $b \in B$ is given by $\{\{b\} \cup (C_b \setminus F) : F \text{ finite}\}$,
- K is a compactification of $A \cup B$.

A brief summary of further steps

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   - $x \in (A \times 2^\omega) \cup \{c\}$ gives a unique point in $\mathbb{L}$,
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Fact

$$w(\mathbb{L}) = w(K) = |A \cup B|.$$

How to get such AD family?

Recall

$\langle C_b : b \in B\rangle \subset \mathcal{P}(A)$ is an AD family. There is a function $D : B \to \omega^{2^{<\omega}}$ st

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In our construction, each $A_t \cong \omega_1$ and $B = \mathrm{Lim}(\omega_1)$.
We fix a function D and $\Delta : B \to \omega^{(2^{<\omega}) \times \omega}$.

We construct $\langle C_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ with $C_\alpha \subseteq \alpha$.

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$$C_n = C'_n \setminus \bigcup_{k<n} C'_k.$$

Fix a bijective $e_\alpha : \alpha \to \omega$. Let

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Observation

If, for t's forming ∞-many levels of $2^{<\omega}$ for ∞-many n $\Delta(\alpha)(t,n)$ is larger than some (e_α-image of) $a \in C_n \cap A_t$ with $s(a) = D(\alpha)(t)$, then α is a $(\star)$-witness for s.

Cardinal Invariants

Relational systems and their evaluations

(A, B, R) with $R \subseteq A \times B$ is called a relational system.

$$\mathfrak{d}(A, B, R) = \min\{\mathcal{B} : \mathcal{B} \subseteq B, \forall_{a \in A} \exists_{b \in \mathcal{B}}\, aRb\}$$

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Examples

- $\mathfrak{d} = \mathfrak{d}(\omega^\omega, \omega^\omega, \geqslant^*)$,
- $\mathfrak{b} = \mathfrak{d}(\omega^\omega, \omega^\omega, \not\leqslant^*)$,
- $\mathrm{non}(\mathcal{M}) = \mathfrak{d}(\mathcal{M}, \mathbb{R}, \not\ni) = \mathfrak{d}(\mathcal{M} \cap F_\sigma, \mathbb{R}, \not\ni) = \mathfrak{d}(\omega^\omega, \omega^\omega, =^\infty)$

Diamonds

$\diamondsuit$

There is $\Delta : \omega_1 \to 2^{<\omega_1}$ so that for any $\delta \in 2^{\omega_1}$ the set

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$\diamondsuit(A, B, R)$ [Moore, Hrušák, and Džamonja 2004]

For any (sufficiently definable) $F : 2^{<\omega_1} \to A$, there is $\Delta : \omega_1 \to B$ so that for any $\delta \in 2^{\omega_1}$ the set

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$\diamondsuit(A, B, R) \implies \mathfrak{d}(A, B, R) \leqslant \omega_1$
in many classical models the converse is also true.

Our oracle

$\Diamond(\mathrm{non}(\mathcal{M}))$ on a slightly peculiar space

For any $F : 2^{<\omega_1} \to \mathcal{M}(\omega^{2^{<\omega}} \times \omega^{2^{<\omega} \times \omega})$, there are $D : \omega_1 \to \omega^{2^{<\omega}}$ and $\Delta : \omega_1 \to \omega^{2^{<\omega} \times \omega}$ so that for any $\delta \in 2^{\omega_1}$ the set

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Let $\gamma \in 2^\alpha$ encode $\langle C_\delta : \delta \in \mathrm{Lim}(\alpha) \rangle$ and $s : A \cap \alpha \to \omega$. Let C_n as before – disjoint, enumerated version of C_δ.

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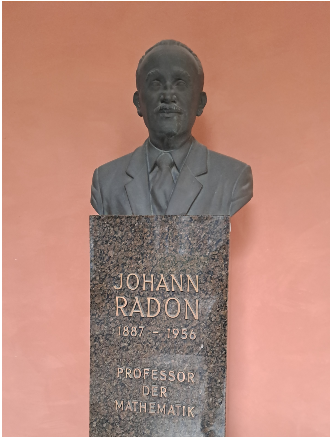
$$F(\delta) = \{\langle x, y\rangle \in \omega^{2^{<\omega}} \times \omega^{2^{<\omega} \times \omega} : \forall^\infty_l \exists_{t \in 2^l} \forall^\infty_n \forall_{a \in A_t \cap C_n} s(a) \neq x(t) \text{ or } y(t,n) < e_\alpha(a)\}$$

Thank you!



Radon in Vienna

Nikodym in Kraków

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- Avilés, Antonio and Piotr Koszmider (2013). "A continuous image of a Radon-Nikodym compact space which is not Radon-Nikodym". In: *Duke Math. J.* 162.12.
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