Game-theoretic variants of splitting number

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Winter School in Abstract Analysis 2025

Joint work with Jorge Antonio Cruz Chapital, Tatsuya Goto and Yusuke Hayashi

Table

1. Splitting* game and splitting** game
2. Forcing notions $\mathbb{P}^*$ and $\mathbb{P}^{**}$
3. UF-limits

splitting* game

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Table: splitting* game.

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Player II has a winning strategy

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Thus, Player I does not have a winning strategy.

What if Player II's play is restricted?

Restricted games

Let $\mathcal{A} \subseteq 2^\omega$. The splitting*[splitting**] game with respect to $\mathcal{A}$ is a game following the same rule as the splitting*[splitting**] game, but the winning condition is replaced by:

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$$\mathfrak{s}^*_{\mathrm{II}} := \min \left\{ |\mathcal{A}| : \begin{array}{l} \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ Player II has a winning strategy} \\ \text{for the splitting* game with respect to } \mathcal{A} \end{array} \right\},$$

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Define $\mathfrak{s}^{**}_{\mathrm{II}}$ and $\mathfrak{s}^{**}_{\mathrm{I}}$ similarly for the splitting** game.

$\mathfrak{s}_{\mathrm{II}}^{\circ} = \min\{|\mathcal{A}| :$ P. II has a w.s. for the splitting° game w.r.t. $\mathcal{A}\}$
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Lemma

$\mathfrak{s} \leq \mathfrak{s}^*_{\mathrm{I}}$.

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Lemma

$\mathfrak{s} \leq \mathfrak{s}^*_{\mathrm{I}}$.

Proof. Let $\mathcal{A}$ witness $\mathfrak{s}^*_{\mathrm{I}}$. Given an infinite x, let σ be a strategy such that $\sigma * y = x$ for any y. σ is not winning over $\mathcal{A}$, so some $y \in \mathcal{A}$ splits $\sigma * y = x$. Since x was arbitrary, $\mathcal{A}$ is a splitting family. □

ZFC results

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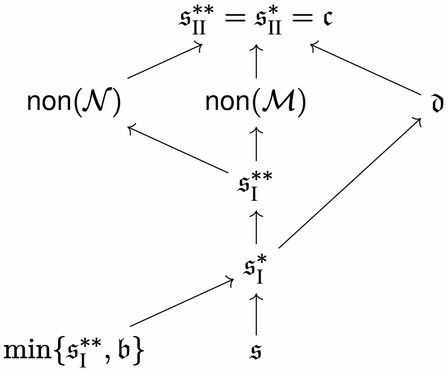

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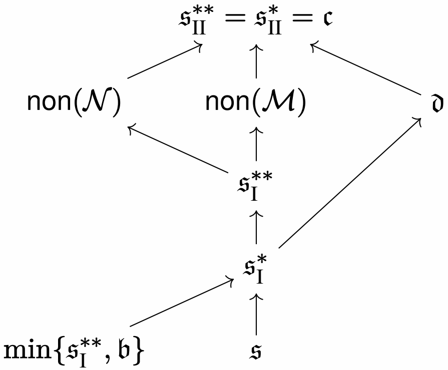

Question

Does $\mathfrak{s}_{\mathrm{I}}^{**} \leq \mathfrak{d}$ hold?

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Theorem (Cruz, Goto, Hayashi and Y.)

$\mathfrak{s} < \mathfrak{s}_{\mathrm{I}}^* < \mathfrak{s}_{\mathrm{I}}^{**}$ consistently holds.

The "$\mathfrak{s} <$" part is obtained by application of their method by Goldstern, Kellner, Mejía and Shelah [GKMS21] of preserving splitting families through the forcing iteration, so we focus on $\mathfrak{s}_{\mathrm{I}}^* < \mathfrak{s}_{\mathrm{I}}^{**}$.

Enjoy $\mathrm{Con}(\mathfrak{s}_{\mathrm{I}}^* < \mathfrak{s}_{\mathrm{I}}^{**})$ more!

Enjoy the consistency of $\mathfrak{s}_{\mathrm{I}}^* < \mathfrak{s}_{\mathrm{I}}^{**}$ more!

Enjoy $\mathrm{Con}(\mathfrak{s}^*_{\mathrm{I}} < \mathfrak{s}^{**}_{\mathrm{I}})$ more!

Enjoy the consistency of $\mathfrak{s}^*_{\mathrm{I}} < \mathfrak{s}^{**}_{\mathrm{I}}$ more! Recall:
$\mathfrak{s}^*_{\mathrm{I}} = \min\{|\mathcal{A}|$: P. I has no w.s. for the splitting* game w.r.t. $\mathcal{A}\}$
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Assume:

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2. $\mathfrak{s}_{\mathrm{I}}^* < \mathfrak{s}_{\mathrm{I}}^{**}$ holds and $\mathcal{A} \subseteq 2^\omega$ is a witness of $\mathfrak{s}_{\mathrm{I}}^*$.

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Since $|\mathcal{A}| = \mathfrak{s}_{\mathrm{I}}^{*} < \mathfrak{s}_{\mathrm{I}}^{**}$, there is a winning strategy σ for the *splitting*** game w.r.t. $\mathcal{A}$, so:

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4. I follow the strategy σ.
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You can beat me, because $\mathcal{A}$ witnesses $\mathfrak{s}_{\mathrm{I}}^*$, so my σ is not a w.s. for this *splitting** game w.r.t. $\mathcal{A}$.

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1. Splitting* game and splitting** game
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3. UF-limits

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Thus,
$\mathfrak{s}_{\mathrm{I}}^* = \min\{|\mathcal{A}| : \mathcal{A} \subseteq 2^\omega,\ (\neg\exists\sigma \in \mathrm{Str})\ (\forall y \in \mathcal{A})\ (y \lhd_* \sigma)\}$, and
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We introduce the poset $\mathbb{P}^{**}$ which generically adds a winning strategy for the splitting** game and hence increases $\mathfrak{s}_{\mathrm{I}}^{**}$:

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Note that for $x, y \in 2^\omega \setminus \mathbb{O}$, if $x \leq^* y$ then $x \subseteq^* y$, i.e., x is almost included by y and particularly x is not split by y.

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For any $y \in 2^\omega$, $\Vdash_{\mathbb{P}^{**}} y \vartriangleleft_{**} \sigma_G$. Hence, $\mathbb{P}^{**}$ increases $\mathfrak{s}_{\mathrm{I}}^{**}$ by finite support iteration (Note that $\mathbb{P}^{**}$ is σ-centered and thus ccc).

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We showed $\mathbb{P}^{**}$ increases $\mathfrak{s}_{\mathrm{I}}^{**}$.

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We focus on the first item in this talk.

UF-limits

An uf denotes a non-principal ultrafilter on ω in this talk.

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1. Let D be an uf. $Q \subseteq \mathbb{P}$ is D-lim-linked if there exist a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter extending D and a function $\lim^D : Q^\omega \to \mathbb{P}$ such that for any $\bar{q} = \langle q_m : m < \omega \rangle \in Q^\omega$,

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$Q \subseteq \mathbb{P}$ is UF-lim-linked if it is D-lim-linked for any uf D.
2. $\mathbb{P}$ has UF-limits if $\mathbb{P}$ is a union of countably many UF-lim-linked components.

Key Lemma 1: $\mathbb{P}^{**}$ has UF-limits

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$\lim^D \bar{q} := (\sigma, \{y_i^\infty : i < k\})$ does work (details omitted). □

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$Q_{\sigma,k} := \{(\sigma', F) \in \mathbb{P}^{**} : \sigma' = \sigma, |F| \leq k\}$ is UF-lim-linked for $\sigma \in \mathrm{FinStr}$ and $k < \omega$. In particular, $\mathbb{P}^{**}$ has UF-limits.

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$\bar{q} = \langle q_m := (\sigma, F_m = \{y_i^m : i < k\}) : m < \omega \rangle \in (Q_{\sigma,k})^\omega$.
For $i < k$, define $y_i^\infty \in 2^\omega$ by for $n < \omega$ and $j \in 2$:

$$y_i^\infty(n) = j :\Leftrightarrow \{m < \omega : y_i^m(n) = j\} \in D.$$

$\lim^D \bar{q} := (\sigma, \{y_i^\infty : i < k\})$ does work (details omitted). □

Remark

In the case of $\mathbb{P}^* = \{(\sigma, F) \in \mathbb{P}^{**} : F \subseteq 2^\omega \setminus \mathbb{0}\}$, even if all y_i^m are not in $\mathbb{0}$, y_i^∞ might be in $\mathbb{0}$ and hence the same proof does not work.

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Definition

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Lemma

Let X be a compact (Hausdorff) space and D an uf. Then, every countable sequence in X has a (unique) D-limit.

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Lemma

Let X be a compact (Hausdorff) space and D an uf. Then, every countable sequence in X has a (unique) D-limit.

Proof of existence. Assume $\langle x_m : m < \omega \rangle$ has no D-limits. Then, each $x \in X$ has some open neighborhood U_x such that $A_x := \{m < \omega : x_m \notin U_x\} \in D$. Since $\bigcup_{x \in X} U_x = X$, some finite $F \subseteq X$ satisfies $\bigcup_{x \in F} U_x = X$ by compactness. Therefore, $\bigcap_{x \in F} A_x = \{m < \omega : x_m \notin \bigcup_{x \in F} U_x = X\} \in D$ is non-empty, a contradiction. □

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Actually, each y_i^∞ is the D-limit of $\bar{y}_i := \langle y_i^m : m < \omega \rangle$ in Cantor space 2^ω, which is compact.

In comparison, in the case of $\mathbb{P}^* = \{(\sigma, F) \in \mathbb{P}^{**} : F \subseteq 2^\omega \setminus \mathbb{0}\}$, $2^\omega \setminus \mathbb{0}$ is not compact!

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Question

Can we find another (non-artificial) example of a pair of two numbers such that their difference lies in whether their corresponding forcing notions have compactness or not, and consequently they are consistently different?

References I

[CGHY24] Jorge Antonio Cruz Chapital, Tatsuya Goto, Yusuke Hayashi, and Takashi Yamazoe. Game-theoretic variants of splitting number. Preprint, arXiv:2412.19556, 2024.

[GKMS21] Martin Goldstern, Jakob Kellner, Diego A. Mejía, and Saharon Shelah. Preservation of splitting families and cardinal characteristics of the continuum. *Israel Journal of Mathematics*, 246(1):73–129, 2021.

[GMS16] Martin Goldstern, Diego A. Mejía, and Saharon Shelah. The left side of Cichoń's diagram. *Proceedings of the American Mathematical Society*, 144(9):4025-4042, 2016.