Strongly increasing sequences of functions

Chris Lambie-Hanson

Institute of Mathematics
Czech Academy of Sciences

Winter School in Abstract Analysis
Hejnice

31 January 2025

Joint work with Paul Larson.

Motivating questions

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Motivating questions

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Motivating questions

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Both questions are known to have positive answers if $\aleph_2$ is replaced by $\aleph_1$.

Motivating questions

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Both questions are known to have positive answers if $\aleph_2$ is replaced by $\aleph_1$. If $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ holds and there exists a Woodin cardinal, then there is a forcing extension via stationary tower forcing in which $(\aleph_{\omega+1})^V = \aleph_2$.

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

- For all $n < \omega$, we say that $f <_n g$ if $f(m) < g(m)$ for all $m \in \omega \setminus n$.

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

- For all $n < \omega$, we say that $f <_n g$ if $f(m) < g(m)$ for all $m \in \omega \setminus n$.
- We say that $f <^* g$ if there exists $n < \omega$ such that $f <_n g$.

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

- For all $n < \omega$, we say that $f <_n g$ if $f(m) < g(m)$ for all $m \in \omega \setminus n$.
- We say that $f <^* g$ if there exists $n < \omega$ such that $f <_n g$.

Definition

Let δ be an ordinal and $\vec{f} = \langle f_\alpha \mid \alpha < \delta \rangle$ a sequence of elements of ${}^\omega\mathrm{Ord}$. We say that $\vec{f}$ is *strongly increasing* if

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

- For all $n < \omega$, we say that $f <_n g$ if $f(m) < g(m)$ for all $m \in \omega \setminus n$.
- We say that $f <^* g$ if there exists $n < \omega$ such that $f <_n g$.

Definition

Let δ be an ordinal and $\vec{f} = \langle f_\alpha \mid \alpha < \delta \rangle$ a sequence of elements of ${}^\omega\mathrm{Ord}$. We say that $\vec{f}$ is *strongly increasing* if

1. $\vec{f}$ is $<^*$-increasing; and

Increasing sequences

Suppose that $f, g : \omega \to \mathrm{Ord}$.

- For all $n < \omega$, we say that $f <_n g$ if $f(m) < g(m)$ for all $m \in \omega \setminus n$.
- We say that $f <^* g$ if there exists $n < \omega$ such that $f <_n g$.

Definition

Let δ be an ordinal and $\vec{f} = \langle f_\alpha \mid \alpha < \delta \rangle$ a sequence of elements of ${}^\omega\mathrm{Ord}$. We say that $\vec{f}$ is *strongly increasing* if

1. $\vec{f}$ is $<^*$-increasing; and
2. for every limit ordinal $\beta < \delta$, there is a club C_β in β and an $n_\beta < \omega$ such that $f_\alpha <_{n_\beta} f_\beta$ for all $\alpha \in C_\beta$.

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.
2. There is a strongly increasing sequence of length $\aleph_{\omega+1}$ in $\prod_{k<\omega} \omega_k$.

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.
2. There is a strongly increasing sequence of length $\aleph_{\omega+1}$ in $\prod_{k<\omega} \omega_k$.
3. (Shelah) If $4 \leq m < \omega$ and $\gamma < \omega_m$, then there is no strongly increasing sequence of length ω_m in ${}^{\omega}\gamma$.

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.
2. There is a strongly increasing sequence of length $\aleph_{\omega+1}$ in $\prod_{k<\omega} \omega_k$.
3. (Shelah) If $4 \leq m < \omega$ and $\gamma < \omega_m$, then there is no strongly increasing sequence of length ω_m in ${}^{\omega}\gamma$.

Corollary

Suppose that $V \subseteq W$ are inner models of ZFC *and $4 \leq m < \omega$. Then $(\aleph_{\omega+1})^V \neq (\aleph_m)^W$.*

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.
2. There is a strongly increasing sequence of length $\aleph_{\omega+1}$ in $\prod_{k<\omega} \omega_k$.
3. (Shelah) If $4 \leq m < \omega$ and $\gamma < \omega_m$, then there is no strongly increasing sequence of length ω_m in ${}^{\omega}\gamma$.

Corollary

Suppose that $V \subseteq W$ are inner models of ZFC *and $4 \leq m < \omega$. Then $(\aleph_{\omega+1})^V \neq (\aleph_m)^W$.*

Proof.

Otherwise, use (2) to fix in V a strongly increasing sequence $\vec{f}$ in $\prod_{n<\omega} \omega_n$ of length $\aleph_{\omega+1}$.

Basic facts

1. There is a strongly increasing sequence of length ω_1 in ${}^{\omega}\omega$.
2. There is a strongly increasing sequence of length $\aleph_{\omega+1}$ in $\prod_{k<\omega} \omega_k$.
3. (Shelah) If $4 \leq m < \omega$ and $\gamma < \omega_m$, then there is no strongly increasing sequence of length ω_m in ${}^{\omega}\gamma$.

Corollary

Suppose that $V \subseteq W$ are inner models of ZFC *and $4 \leq m < \omega$. Then $(\aleph_{\omega+1})^V \neq (\aleph_m)^W$.*

Proof.

Otherwise, use (2) to fix in V a strongly increasing sequence $\vec{f}$ in $\prod_{n<\omega} \omega_n$ of length $\aleph_{\omega+1}$. In W, $\vec{f}$ is a strongly increasing sequence of length ω_m in ${}^{\omega}\gamma$, where $\gamma = (\aleph_\omega)^V < \omega_m$, contradicting (3). $\square$

Motivating questions again

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Both questions are known to have positive answers if ω_2 is replaced by ω_1 and negative answers if it is replaced by ω_m for $4 \leq m < \omega$.

Motivating questions again

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Both questions are known to have positive answers if ω_2 is replaced by ω_1 and negative answers if it is replaced by ω_m for $4 \leq m < \omega$. A positive answer to either question would entail the consistency of the existence of a strongly increasing sequence of length ω_2 in ${}^\omega\omega_1$.

Motivating questions again

Question (Bukovský–Copláková-Hartová)

Can there exist a pair of inner models of ZFC $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$*?*

Question

Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent?*

Both questions are known to have positive answers if ω_2 is replaced by ω_1 and negative answers if it is replaced by ω_m for $4 \leq m < \omega$. A positive answer to either question would entail the consistency of the existence of a strongly increasing sequence of length ω_2 in ${}^\omega\omega_1$.

Question

Is there consistently a strongly increasing sequence of length ω_2 *in* ${}^\omega\omega_1$ *(or* ${}^\omega\omega$*)?*

An answer

An answer

Theorem (Larson–LH)

Suppose that $V = \mathrm{L}(\mathbb{R})$ *and* AD *holds (or* $V = \mathrm{L}(A, \mathbb{R})$ *for some* $A \subseteq {}^{\omega}\omega$ *and* AD^{+} *holds). Then there is a forcing extension satisfying "*ZFC+ *there exists a strongly increasing sequence of length* ω_2 *in* ${}^{\omega}\omega$*".*

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing.

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}_{\omega_1}^{M}$;

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}^M_{\omega_1}$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^\omega\omega$ whose length is a successor ordinal less than ω_2^M;

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}_{\omega_1}^M$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^\omega\omega$ whose length is a successor ordinal less than ω_2^M;
- $a \in \mathcal{P}(\omega_1)^M$ is such that, for some $x \in \mathcal{P}(\omega)^M$, we have $\omega_1^M = \omega_1^{L[a,x]}$.

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}_{\omega_1}^M$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^\omega\omega$ whose length is a successor ordinal less than ω_2^M;
- $a \in \mathcal{P}(\omega_1)^M$ is such that, for some $x \in \mathcal{P}(\omega)^M$, we have $\omega_1^M = \omega_1^{L[a,x]}$.

We set $(N, H, b) < (M, F, a)$ if there exists an iteration $j : (M, \mathsf{NS}_{\omega_1}^M) \to (M^*, \mathsf{NS}_{\omega_1}^{M^*})$ such that $j \in N$,

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}^M_{\omega_1}$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^\omega\omega$ whose length is a successor ordinal less than ω_2^M;
- $a \in \mathcal{P}(\omega_1)^M$ is such that, for some $x \in \mathcal{P}(\omega)^M$, we have $\omega_1^M = \omega_1^{L[a,x]}$.

We set $(N, H, b) < (M, F, a)$ if there exists an iteration $j : (M, \mathsf{NS}^M_{\omega_1}) \to (M^*, \mathsf{NS}^{M^*}_{\omega_1})$ such that $j \in N$, $j(a) = b$,

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}^M_{\omega_1}$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^{\omega}\omega$ whose length is a successor ordinal less than ω_2^M;
- $a \in \mathcal{P}(\omega_1)^M$ is such that, for some $x \in \mathcal{P}(\omega)^M$, we have $\omega_1^M = \omega_1^{L[a,x]}$.

We set $(N, H, b) < (M, F, a)$ if there exists an iteration $j : (M, \mathsf{NS}^M_{\omega_1}) \to (M^*, \mathsf{NS}^{M^*}_{\omega_1})$ such that $j \in N$, $j(a) = b$, $\mathsf{NS}^{M^*}_{\omega_1} = \mathsf{NS}^N_{\omega_1} \cap M^*$,

A sketch of the forcing

We force over $\mathrm{L}(\mathbb{R})$ with a variant $\mathbb{P}$ of Woodin's $\mathbb{P}_{\max}$ forcing. More precisely, conditions are all triples of the form (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ that is iterable with respect to $\mathsf{NS}_{\omega_1}^M$;
- F is, in M, a (very) strongly increasing sequence of functions in ${}^{\omega}\omega$ whose length is a successor ordinal less than ω_2^M;
- $a \in \mathcal{P}(\omega_1)^M$ is such that, for some $x \in \mathcal{P}(\omega)^M$, we have $\omega_1^M = \omega_1^{L[a,x]}$.

We set $(N, H, b) < (M, F, a)$ if there exists an iteration $j : (M, \mathsf{NS}_{\omega_1}^M) \to (M^*, \mathsf{NS}_{\omega_1}^{M^*})$ such that $j \in N$, $j(a) = b$, $\mathsf{NS}_{\omega_1}^{M^*} = \mathsf{NS}_{\omega_1}^N \cap M^*$, and $j(F)$ is an initial segment of H.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.
- $\mathbb{P}$ is countably closed.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.
- $\mathbb{P}$ is countably closed.
- Let G be $\mathbb{P}$-generic over V. Then

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.
- $\mathbb{P}$ is countably closed.
- Let G be $\mathbb{P}$-generic over V. Then
    - $V[G] \models \mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.
- $\mathbb{P}$ is countably closed.
- Let G be $\mathbb{P}$-generic over V. Then
  - $V[G] \models \mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$.
  - Let $A_G = \bigcup\{a \mid (M, F, a) \in G\}$. Then for each $p = (M, F, a) \in G$, there is a unique iteration $j_{p,G}$ of $(M, \mathrm{NS}^M_{\omega_1})$ sending a to A_G.

AD (or AD^+) implies the following facts about $\mathbb{P}$.

- For each $x \in H(\omega_1)$, there are densely many conditions $(M, F, a) \in \mathbb{P}$ such that $x \in H(\omega_1)^M$.
- $\mathbb{P}$ is countably closed.
- Let G be $\mathbb{P}$-generic over V. Then
  - $V[G] \models \mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$.
  - Let $A_G = \bigcup\{a \mid (M, F, a) \in G\}$. Then for each $p = (M, F, a) \in G$, there is a unique iteration $j_{p,G}$ of $(M, \mathrm{NS}^M_{\omega_1})$ sending a to A_G.
  - $\bigcup\{j_{p,G}(F) \mid p = (M, F, a) \in G\}$ is a (very) strongly increasing sequence in ${}^\omega\omega$ of length ω_2.

More questions

Question

Does the existence of a strongly increasing sequence in ${}^{\omega}\omega$ of length ω_2 have any large cardinal strength?

More questions

Question

Does the existence of a strongly increasing sequence in ${}^{\omega}\omega$ *of length* ω_2 *have any large cardinal strength?*

Question

Is there consistently a strongly increasing sequence in ${}^{\omega}\omega$ *of length* ω_3*?*

Thank you!