Inner models from extended logics

(All three lectures in one file)

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Inner model from extended logics

Lecture 1: Extended logics, inner models, examples, *L*-tameness.

Lecture 2: Stationary logic, a Completeness Theorem, Club Determinacy, Applications.

Lecture 3: Second order logic, HOD.

The tutorial is based on

- **Inner Models from Extended Logics: Part 1**, J. Kennedy, M. Magidor and J.V. Journal of Mathematical Logic (2021).
- **Inner Models from Extended Logics: Part 2**, J. Kennedy, M. Magidor and J.V. Journal of Mathematical Logic. (to appear)
- Also relevant: **Closed and unbounded classes and the Härtig quantifier model.**, Ph. Welch, J. Symb. Log. (2022).

We will learn in this first lecture:

- A new construction of a whole family of inner models.
- Why some of them are not really new.
- Why and how some of them extend known inner models.

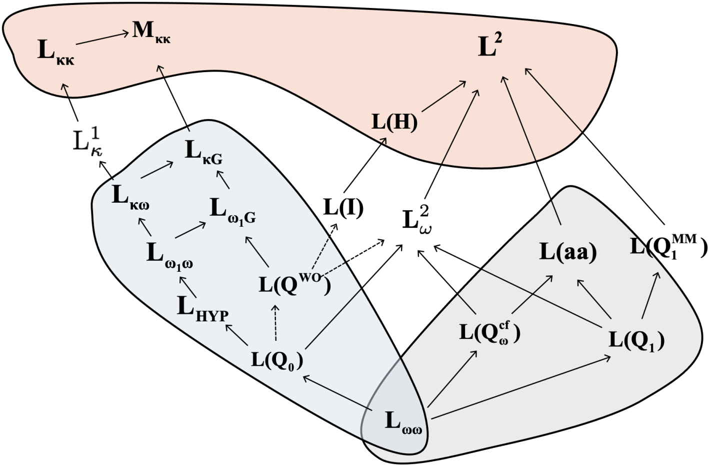

Figure: Map of extended logics

Possible desirable attributes of extended logics

- Axiomatizable.
- Downward Löwenheim-Skolem Theorem (in some form).
- Compactness Theorem (in some form).
- Can express interesting mathematical properties.
- Abstract Model Theory [1].

Common inner models

- Cumulative hierarchy V.
- Constructible sets L.
- Hereditarily ordinal definable sets HOD.
- $L[\mu]$, $L[\vec{E}]$
- $L(\mathbb{R})$
- Chang model $C_{\omega_1\omega_1}$.

Possible desirable attributes of inner models

- Forcing absolute.
- Support large cardinals.
- Arise "naturally".
- Decide questions such as CH.
- Satisfy Axiom of Choice.

No (very) large cardinals in L.

Scott 1961 [12].

- Suppose $V = L$ and κ is (the least) MC with n.u.f. F.
- Let $N = V^\kappa$.
- Define $f \sim g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$.
- Define $[f]\ E\ [g] \iff \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in U$.
- Let $(M, \in)$ be the Mostowski collapse of (N, E).
- Let $i(a) = [c_a]$.
- $i : V \to M$ elementary, $i(\kappa) > \kappa$.
- $M \models$ "$i(\kappa)$ is the least MC".
- But $M = V$, a contradiction.

Project: Inner models from extended logics

- Replace first order logic by one of the logics in the Map-of-Logics in order to obtain new inner models with desirable properties.
- The inner model $C(\mathcal{L}^*)$ arises from Gödel's L by replacing first order logic $\mathcal{L}_{\omega\omega}$ by an extension $\mathcal{L}^*$ of $\mathcal{L}_{\omega\omega}$.

The inner model L

In 1940 Gödel introduced L [3].

$$\begin{cases} L_0 & = & \emptyset \\ L_\nu & = & \bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu \\ L_{\alpha+1} & = & \{X \subseteq L_\alpha : X \text{ is } \mathcal{L}_{\omega\omega}\text{-definable over } L_\alpha \\ & & \text{i.e. } X = \{a \in L_\alpha : L_\alpha \models \varphi(a,\vec{b})\} \\ & & \text{for some } \varphi(x,\vec{y}) \in \mathcal{L}_{\omega\omega} \text{ and some } \vec{b} \in L_\alpha\} \\ L & = & \bigcup_\alpha L_\alpha. \end{cases}$$

Theorem

$L \models ZFC$.

The definition of the inner model $C(\mathcal{L}^*)$

$$\begin{cases} L'_0 & = & \emptyset \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \\ L'_{\alpha+1} & = & \{X \subseteq L'_\alpha : X \text{ is } \mathcal{L}^*\text{-definable over } L'_\alpha \\ & & \text{i.e. } X = \{a \in L'_\alpha : L'_\alpha \models \varphi(a, \vec{b})\} \\ & & \text{for some } \varphi(x, \vec{y}) \in \mathcal{L}^* \text{ and some } \vec{b} \in L'_\alpha\} \\ C(\mathcal{L}^*) & = & \bigcup_\alpha L'_\alpha. \end{cases}$$

Theorem

If $\mathcal{L}^*$ has "nice syntax" (e.g. arises from first order logic by adding a finite number of generalized quantifiers), then $C(\mathcal{L}^*) \models ZFC$.

Measuring the strengths of logics

- $\mathcal{L}^* \leq \mathcal{L}^+$ if $\mathcal{L}^* \subseteq \mathcal{L}^+$.
- $\mathcal{L}^* \leq' \mathcal{L}^+$ if $C(\mathcal{L}^*) \subseteq C(\mathcal{L}^+)$.
- A set theoretic perspective to the strength of logics.

What if we don't get anything new...

Definition

A logic $\mathcal{L}^*$ is *L*-tame, if $C(\mathcal{L}^*) = L$.

Can we characterize *L*-tame logics? Does *L*-tameness have model theoretic content?

Tameness of $L(Q_0)$

$Q_0 x\varphi(x,\vec{b}) \iff$ there are infinitely many x satisfying $\varphi(x,\vec{b})$.

Theorem ([6])

$C(\mathcal{L}(Q_0)) = L$.

Lemma

Suppose μ is an ordinal, and $A \subseteq \mu$ such that $A \in L_\kappa$, $\kappa > \mu$. If there is a one-one $f : \omega \to A$, then there is such a function f in L_κ.

Proof.

Suppose there isn't. Since L_κ satisfies AC, there is $n < \omega$ and one-one $g : A \to n$ in L such that $g \in L_\kappa$. But such a g is a one-one $A \to n$ also in V, contradicting the existence of f in V. □

We use induction on α to prove that $L'_\alpha \subseteq L$. Suppose $L'_\alpha \subseteq L$, and hence $L'_\alpha \in L_\gamma$ for some γ. We show that $L'_{\alpha+1} \subseteq L$. Suppose $X \in L'_{\alpha+1}$. Then X is of the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\},$$

where $\varphi(x, \vec{y}) \in \mathcal{L}(Q_0)$ and $\vec{b} \in L'_\alpha$. For simplicity we suppress the mention of $\vec{b}$.

Since we can use induction on φ, the only interesting case is

$$X = \{a \in L'_\alpha : \text{ There is a one-one } f : \omega \to X_a\},$$

where (by ind. hyp.)

$$X_a = \{c \in L'_\alpha : (L'_\alpha, \in) \models \psi(c, a)\} \in L_\kappa,$$

for some $\kappa > \gamma$.

Now the Lemma implies

$$X = \{a \in L'_\alpha : \text{ There is a one-one } f : \omega \to X_a \text{ in } L_\kappa\},$$

Finally,

$$X = \{a \in L_\kappa : (L_\kappa, \in) \models \text{“} a \in L'_\alpha \wedge \\ \text{There is a one-one } f : \omega \to L'_\alpha \text{ such that for all } n \\ \psi(x, f(n)) \text{ is true when relativized to } L'_\alpha\text{”}\}.$$

Thus $X \in L_{\kappa+1}$.

More generally...

$$Q_\alpha x \varphi(x, \vec{b})$$

$$\iff$$

there are at least $\aleph_\alpha$ many x satisfying $\varphi(x, \vec{b})$.

- The same proof gives L-tameness of $\mathcal{L}(Q_{\alpha_0}, \ldots, Q_{\alpha_n})$ for all $\alpha_0, \ldots, \alpha_n$.
- $Q_{\{\alpha_1, \ldots, \alpha_n\}}$ says the cardinality is one of $\{\aleph_{\alpha_1}, \ldots, \aleph_{\alpha_n}\}$. This is also L-tame.

Tameness of the Magidor-Malitz logic $L(Q_0^{MM,2})$

$Q_0^{MM,2}xy\varphi(x,y,\vec{b}) \iff$
there is an infinite set X such that $\forall x,y \in X\varphi(x,y,\vec{b})$.

Theorem ([6])
$C(Q_0^{MM,2}) = L$.

Lemma
Suppose μ is an ordinal, and $A \in L_\kappa$, $\kappa > \mu$, such that $A \subseteq [\mu]^2$. If there is an infinite B such that $[B]^2 \subseteq A$, then there is such a set B in L_κ.

Proof.
Blackboard! □

Tameness of the Magidor-Malitz quantifier, assuming $0^\sharp$

$Q_1^{MM} xy\varphi(x, y, \vec{b}) \iff$
there is an uncountable set X such that $\forall x, y \in X \varphi(x, y, \vec{b})$.

This is not L-tame in general, but:

Theorem ([6])

If $0^\sharp$ exists, then $C(Q_1^{MM}) = L$.

The story of $0^\sharp$

"$0^\sharp$ exists" means and implies that there is a club class I of ordinals such that

1. $L \models \varphi(\gamma_1, ..., \gamma_n) \leftrightarrow \varphi(\gamma'_1, ..., \gamma'_n)$ whenever $\gamma_1 < ... < \gamma_n, \gamma'_1 < ... < \gamma'_n$ are in I and $\varphi(x_1, ..., x_n)$ is a first order formula of set theory.
2. $L_\gamma \prec L$ whenever $\gamma \in I$.
3. If $\gamma \in Lim(I)$, then $\{X \subseteq \gamma : \exists\delta((I \setminus \delta) \cap \gamma \subseteq X)\}$ is an L-ultrafilter.
4. **Rowbottom Property**: Suppose $\gamma \in Lim(I)$. Suppose $C \subseteq [\gamma]^2$, where $C \in L$. Then there is $B \in \mathcal{U}_\gamma$ such that $[B]^2 \subseteq C$ or $[B]^2 \cap C = \emptyset$.

Lemma

Suppose $0^\sharp$ exists, μ is an ordinal, and $A \in L$ such that $A \subseteq [\mu]^2$. If there is an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L.

How does the Lemma helps us to prove the theorem

- We will use induction on α to prove that $L'_\alpha \subseteq L$.
- We suppose $L'_\alpha \subseteq L$, and hence $L'_\alpha \in L_\gamma$ for some canonical indiscernible γ.
- We show that $L'_{\alpha+1} \subseteq L_{\gamma+1}$.
- Suppose $X \in L'_{\alpha+1}$.
- Then X is of the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\},$$

  where $\varphi(x, \vec{y}) \in \mathcal{L}(Q_1^{\text{MM}})$ and $\vec{b} \in L'_\alpha$.
- For simplicity we suppress the mention of $\vec{b}$.

Since we can use induction on φ, the only interesting case is

$$X = \{a \in L'_\alpha : \text{ There is an uncountable set } Y \subseteq L'_\alpha$$

$$\text{such that } [Y]^2 \subseteq X_a\},$$

where (by ind. hyp.)

$$X_a = \{\{c, d\} \in [L'_\alpha]^2 : (L'_\alpha, \in) \models \psi(c, d, a)\} \in L.$$

The Lemma implies

$$X = \{a \in L'_\alpha : \text{ There is an uncountable set } Y \subseteq L'_\alpha,\ Y \in L$$

$$\text{such that } [Y]^2 \subseteq X_a\},$$

Since $L_\gamma \prec L$, we have

$$X = \{a \in L'_\alpha : \text{ There is an uncountable set } Y \subseteq L'_\alpha,\ Y \in L_\gamma$$

$$\text{such that } [Y]^2 \subseteq X_a\},$$

Finally,

$$X = \{a \in L_\gamma : (L_\gamma, \in) \models \text{“} a \in L'_\alpha \wedge$$
$$\text{There is an uncountable } Y \subseteq L'_\alpha \text{ such that for all } x, y \in Y$$
$$\psi(x, y, a) \text{ is true when relativized to } L'_\alpha\text{”}\} \in L_{\gamma+1}.$$

Now the proof of the lemma

- Want to prove: If there is an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L.
- W.l.o.g. $|B| = \aleph_1$, say $B = \{\delta_i : i < \omega_1\}$ in increasing order.
- Let I be the canonical closed unbounded class of indiscernibles for L. Let, for simplicity, $\delta_i = \tau_i(\alpha^i)$, where $\alpha^i \in I$.
- W.l.o.g., τ_i is a fixed term τ.
- W.l.o.g. the mapping $i \mapsto \alpha^i$ is strictly increasing in i.
- Let $\gamma = \sup\{\alpha^i : i < \omega_1\}$. It is a limit point of I because I is a club.

Let

$$C = \{\{\alpha, \alpha'\} \in [\gamma]^2 : \{\tau(\alpha), \tau(\alpha')\} \in A\}. \tag{1}$$

Since $A \in L$, also $C \in L$. By the Rowbottom Property there is $B_0 \in \mathcal{U}_\gamma$ such that

$$[B_0]^2 \subseteq C \text{ or } [B_0]^2 \cap C = \emptyset. \tag{2}$$

Claim: $[B_0]^2 \subseteq C$.

To prove this suppose $[B_0]^2 \cap C = \emptyset$. Since $B_0 \in \mathcal{U}_\gamma$, there is $\xi < \gamma$ such that $(I \setminus \xi) \cap \gamma \subseteq B_0$. We can now find $i, j < \omega_1$ such that

$$\xi < \alpha^i < \gamma, \xi < \alpha^j < \gamma.$$

Then since by the choice of B,

$$\tau(\alpha^i), \tau(\alpha^j) \in B,$$

and $[B]^2 \subseteq A$, we have

$$\{\tau(\alpha^i), \tau(\alpha^j)\} \in A.$$

Hence

$$\{\alpha^i, \alpha^j\} \in C \tag{3}$$

contrary to the assumption $[B_0]^2 \cap C = \emptyset$. We have proved the claim.

So we know $[B_0]^2 \subseteq C$. Now we define

$$B^* = \{\tau(\alpha) : \alpha \in B_0\}. \tag{4}$$

Then $B^* \in L, |B^*| = \aleph_1$.

Claim: $[B^*]^2 \subseteq A$.

Proof of the Claim: Suppose $\{\tau(\alpha), \tau(\alpha')\} \in [B^*]^2$, where $\{\alpha, \alpha'\} \in [B_0]^2$. Thus $\{\alpha, \alpha'\} \in C$ i.e. $\{\tau(\alpha), \tau(\alpha')\} \in A$ and we are done.
Lemma proved.

Similarly, for "there is an uncountable branch"-quantifier.

Consistently, $C(Q_1^{\mathrm{MM}}) \neq L$.

A case of non-tameness

Shelah [13] introduced:

$$Q^{\mathrm{cof}}_{\omega} xy\varphi(x, y, \vec{b}) \iff$$

$$\varphi(x, y, \vec{b}) \text{ defines a linear order of countable cofinality.}$$

Define $C^* = C(\mathcal{L}(Q^{\mathrm{cof}}_{\omega}))$.
Not L-tame:

Theorem ([6])

If $0^\sharp$ exists, then $0^\sharp \in C^*$, so $C^* \neq L$.

- Let I be the canonical set of indiscernibles obtained from $0^\sharp$.
- CLAIM: ordinals ξ which are regular cardinals in L and have cofinality $> \omega$ in V are in I.
- Suppose $\xi \notin I$. Note that $\xi > \min(I)$.
- Let δ be the largest element of $I \cap \xi$.
- Let $\lambda_1 < \lambda_2 < ...$ be an infinite sequence of elements of I above ξ.

- If $\alpha < \xi$, then $\alpha = \tau_{n_\alpha}(\gamma_1, \ldots, \gamma_{m_n}, \lambda_1, \ldots, \lambda_{r_n})$ for some $\gamma_1, \ldots, \gamma_{m_n} \in I \cap \delta$ and some $r_n < \omega$.
- Let us fix n and consider the set $A_n = \{\tau_k(\beta_1, \ldots, \beta_{m_n}, \lambda_1, \ldots, \lambda_{r_n}) : \beta_1, \ldots, \beta_{m_n} < \delta, k < \omega\}$.
- Note that $A_n \in L$ and $|A_n|^L \leq |\delta|^L < \xi$, because ξ is a cardinal in L.
- Let $\eta_n = \sup(A_n)$.

- Since ξ is regular in L, $\eta_n < \xi$.
- Since ξ has cofinality $> \omega$ in V, $\eta = \sup_n \eta_n < \xi$.
- But we have now proved that every $\alpha < \xi$ is below η, a contradiction.
- So we may conclude that necessarily $\xi \in I$.

Let

$$X = \{\xi \in L'_{\aleph_\omega} : (L'_{\aleph_\omega}, \in) \models \text{“}\xi \text{ is regular in } L\text{”} \wedge \neg Q^{cf}_\omega xy(x \in y \wedge y \in \xi)\}$$

Now X is an infinite subset of I and $X \in C(Q^{\mathrm{cof}}_\omega)$. Hence $0^\sharp \in C(Q^{\mathrm{cof}}_\omega)$:

$$0^\sharp = \{\ulcorner\varphi(x_1, \ldots, x_n)\urcorner : (L_{\aleph_\omega}, \in) \models \varphi(\gamma_1, \ldots, \gamma_n)$$

$$\text{for some } \gamma_1 < \ldots < \gamma_n \text{ in } X\}.$$

More about C^* later.

A summary of this lecture

- Abstract logics $\mathcal{L}^*$ give rise to inner models $C(\mathcal{L}^*)$.
- Some logics are provably L-tame.
- Some logics (Magidor-Malitz logic) are L-tame assuming $0^\sharp$.
- Some logics (cofinality logic) are not L-tame assuming $0^\sharp$.

Next lecture

- Stationary logic and its inner model $C(\mathrm{aa})$.
- A Completeness Theorem using iterated generic ultrapowers.
- Club Determinacy from a proper class of Woodin cardinals.
- Applications: forcing absoluteness, large cardinals, CH

This lecture

- Stationary logic and its inner model $C(\mathrm{aa})$.
- A Completeness Theorem using iterated generic ultrapowers. (Joint work with B. Velickovic)
- Club Determinacy from a proper class of Woodin cardinals.
- Applications: forcing absoluteness, large cardinals, CH

Stationary logic L(aa)

Suppose we have a model $\mathcal{M}$ and a formula $\varphi(s, \vec{b})$.

I	x_0		x_2		$\dots$
II		x_1		x_3	$\dots$

Player II wins if the set $\{x_0, x_1, x_2, \dots\}$ satisfies $\varphi(s, \vec{b})$ in $\mathcal{M}$.

We write this $\mathcal{M} \models \mathrm{aa}\, s\, \varphi(s, \vec{b})$.

“aa” is short for “almost all”.

Stationary logic L(aa)

$$\text{aa}s\varphi(s,\boldsymbol{a}) \iff \forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \varphi(\{x_0, x_1, x_2, \ldots\}, \boldsymbol{a})$$

$$\iff \{A \in \mathcal{P}_{\omega_1}(M) \ : \ (\mathcal{M}, A) \models \varphi(A, \boldsymbol{a})\}$$

contains a club of countable subsets M.

$$Q_1 x\varphi(x,\boldsymbol{a}) \iff |\{b \in M : \mathcal{M} \models \varphi(b,\boldsymbol{a})\}| \geq \aleph_1$$

$$\iff \neg \text{aa}s\forall y(\varphi(y) \to s(y)).$$

$$Q^{cof}_{\omega} xy\varphi(x,y,\boldsymbol{a}) \iff \text{aa}s\forall x(\exists y\varphi(x,y) \to \exists y(\varphi(x,y) \wedge s(y)))$$

A completeness theorem[1] for $\mathcal{L}(aa)$

- Consider
  (1) φ has a model and
  (2) there is a model (of set theory) for "φ has a model".
- We prove (1) and (2) are equivalent.
- The easy direction is (1) $\Longrightarrow$ (2). Follows from Reflection.
- To prove the other direction we start with a countable model M_0 of "φ has a model $\mathcal{A}$".
- We construct an "$\mathcal{L}(aa)$-absolute" (to be explained) elementary extension M_{ω_1} of M_0.
- Then M_{ω_1} satisfies "φ has a model $j_{\omega_1}(\mathcal{A})$".
- Since M_{ω_1} is "$\mathcal{L}(aa)$-absolute", φ really has a model, and we have (1).

[1] [2], [13]

A completeness theorem for $\mathcal{L}$(aa)

- We build an elementary chain of length ω_1^V.
- $M_0 \underset{j_{01}}{\rightarrow} M_1 \underset{j_{12}}{\rightarrow} M_2 \rightarrow \cdots \rightarrow M_\alpha \underset{j_{\alpha\alpha+1}}{\rightarrow} M_{\alpha+1} \cdots \rightarrow M_{\omega_1}$
- Notation: $j_{0\alpha} = j_\alpha$.

Generic ultrapower

- Let us construct the embedding $M_0 \underset{j_1}{\to} M_1$.
- W.l.o.g. $M_0 \models ZFC_n$.
- Let $\mathbb{P} = (NS^+_{\omega_1})^M$, i.e. the stationary subsets of ω_1^M in the sense of M.
- Let G be $\mathbb{P}$-generic over M_0. As M_0 is countable, G exists in V. Note that G contains all clubs of M.
- Let $N = \{f \in M \mid f : \omega_1^M \to M\}$.
- Define $f \sim g \iff \{\alpha < \omega_1^M : f(\alpha) = g(\alpha)\} \in G$.
- Let $M_1 = N/\sim$ and $j_1(a) = [c_a]$, where $c_a(\alpha) = a$.

A completeness theorem for $\mathcal{L}(aa)$

- (Łoś Lemma) For first order φ we have

$$M_1 \models \varphi([f_1], \ldots, [f_n]) \iff$$

$$\{\alpha \in \omega_1^M \ : \ M_0 \models \varphi(f_0(\alpha), \ldots, f_n(\alpha))\} \in G$$

- $j_1 : M_0 \to M_1$ is elementary.
- $[id] \in j(a)$ if and only if $a \in G$.
- M_1 need not be well-founded.

A completeness theorem for $\mathcal{L}(\mathrm{aa})$

- $j_1(\alpha) =$ "α" i.e. has exactly α predecessors in M_1, when $\alpha < \omega_1$.
- $\sup_{\alpha<\omega_1^M}(j_1(\alpha)) = [id]$
- $j_1(\omega_1^M) > [id]$
- A new element $[id]$ is put to the ω_1 of the model M_1.
- Consequence: $\omega_1^{M_{\omega_1}}$ is $\aleph_1$-like.

Completeness Theorem for L(aa)

- Some hassle arises from the fact that L(aa) has second order variables in addition to first order variables.
- If the domain of the model is ω_1 (or just $\aleph_1$-like with a copy of ω_1 inside, call it E), we can aa-quantify over countable ordinals (or initial segments determined by elements of E in the case of an $\aleph_1$-like ordering with a copy E of ω_1 inside) rather than countable subsets.
- This is because if C is a club of countable subsets of ω_1, then the set $D \subseteq C$ consisting of countable ordinals (or initial segments determined by elements of E in the case of an $\aleph_1$-like ordering with a copy E of ω_1 inside) that are (as sets) in C is also a club as a set of countable ordinals.

Pulling $j_{\omega_1}(\mathcal{A})$ from M_{ω_1} to V

- $M_0 \models$ "$\mathcal{A} \models \varphi$", and therefore $M_{\omega_1} \models$ "$j_{\omega_1}(\mathcal{A}) \models \varphi$"
- We define a model $\mathcal{A}^*$ in V.
- The domain of $\mathcal{A}^*$ is $A^* = \{a \in M_{\omega_1} : M_{\omega_1} \models a \in \omega_1^{M_{\omega_1}}\}$
- $R^{\mathcal{A}^*} = \{(a, b) \in A^* \times A^* \ : \ M_{\omega_1} \models \text{"}j_{\omega_1}(\mathcal{A}) \models R(a, b)\text{"}\}$

The goal

1. Starting point is $\mathcal{A} \models \varphi$, where $\varphi \in L(\mathrm{aa})$.
2. We have reached: $M_{\omega_1} \models$ "$j_{\omega_1}(\mathcal{A}) \models \varphi$".
3. We need to prove by induction on subformulas ψ of φ:

$$(\star) \quad \mathcal{A}^* \models \psi \iff M_{\omega_1} \models \text{“} j_{\omega_1}(\mathcal{A}) \models \psi \text{”}.$$

4. Then, letting ψ be φ, $\mathcal{A}^* \models \varphi$ will follow from (2) and (3) and we are done.
5. As usual, we need to add parameters to $(\star)$ as they arise from quantifiers.
6. To deal with parameters we adopt a lot constant symbols.

Some preliminaries

- Let K be a set of $\aleph_1$ new constant symbols. Assume $K \in M_0$.
- Let $\{S_{\varphi(s,\vec{x}),\vec{c}} : \varphi(s,\vec{x}) \in L(\mathrm{aa}), \vec{c} \in K^{<\omega}\}$ be a splitting in V of ω_1 into ω_1 disjoint stationary sets.
- When we construct the models M_α we make sure that every element of M_α which is in $\omega_1^{M_\alpha}$ is the value of a constant symbol from K.
- That is, we keep expanding the models M_α so that every element of M_α which is in $\omega_1^{M_\alpha}$ is the value of a constant symbol from K.
- Since always the domain of $j_\alpha(\mathcal{A})$ is $\omega_1^{M_\alpha}$, and we consider aa-truth in $j_\alpha(\mathcal{A})$ only, we may drop the parameters, because they are represented by the constant symbols.

A completeness theorem for $\mathcal{L}$(aa)

- Suppose M_{ω_1} satisfies "$j_{\omega_1}(\mathcal{A}) \models \text{aa } s\psi(s)$", and, for simplicity, there are no parameters.
- Then M_0 satisfies "$\mathcal{A} \models \text{aa } s\psi(s)$".
- Hence $\{\alpha < \omega_1^{M_0} : M_0 \models \text{"}\mathcal{A} \models \psi'(\alpha)\text{"}\}$ is a club and therefore is in G_0, whichever way G_0 is chosen. Here $\psi'(\alpha)$ is obtained from $\psi(s)$ by changing everywhere "$s(t)$" to $t < \alpha$.
- Hence $M_1 \models \text{"}\mathcal{A} \models \psi'([id_0])\text{"}$.
- Similarly, $M_{\alpha+1} \models \text{"}j_{\alpha+1}(\mathcal{A}) \models \psi'([id_\alpha])\text{"}$ for all α.
- In the end, $M_{\omega_1} \models \text{"}j_{\omega_1}(\mathcal{A}) \models \psi'(j_{\alpha\omega_1}[id_\alpha])\text{"}$ for all α.
- By Ind. Hyp, $\mathcal{A}^* \models \psi'(j_{\alpha\omega_1}[id_\alpha])$ for all α.
- Hence $\mathcal{A}^* \models \text{aa } s\psi(s)$.

A completeness theorem for $\mathcal{L}(aa)$

- Conversely, suppose M_{ω_1} satisfies "$j_{\omega_1}(\mathcal{A}) \not\models \mathrm{aa} s \psi(s)$".
- Thus M_{ω_1} satisfies "$j_{\omega_1}(\mathcal{A}) \models \mathit{stat}\ s \neg\psi(s)$".
- Then M_0 satisfies $\mathcal{A} \models \mathit{stat}\ s \neg\psi(s)$.
- Hence $S = \{\alpha < \omega_1^{M_0} : M_0 \models \text{"}\mathcal{A} \models \neg\psi'(\alpha)\text{"}\}$ is stationary and therefore we can choose G_0 so that S is in G_0.
- Hence $M_1 \models \text{"}\mathcal{A} \models \neg\psi'([id_0])\text{"}$.
- Similarly, $M_{\alpha+1} \models \text{"}j_{\alpha+1}(\mathcal{A}) \models \neg\psi'([id_\alpha])\text{"}$ for all α. But we choose G_α so that (the corresponding set) S is in G_α only if $\alpha \in S_{\psi(s),\vec{c}}$, where $\vec{c}$ are the constant occurring in $\psi(s)$.
- In the end, $M_{\omega_1} \models \text{"}j_{\omega_1}(\mathcal{A}) \models \neg\psi'(j_{\alpha\omega_1}[id_\alpha])\text{"}$ for all $\alpha \in S_{\psi'(s),\vec{c}}$.
- By Ind. Hyp, $\mathcal{A}^* \models \neg\psi'(j_{\alpha\omega_1}[id_\alpha])$ for all $\alpha \in S_{\psi'(s),\vec{c}}$.
- Hence $\mathcal{A}^* \models \mathit{stat}\ s \neg\psi(s)$.
- Hence $\mathcal{A}^* \not\models \mathrm{aa}\ s\psi(s)$.

The definition of $C(\text{aa}) = C(\mathcal{L}(\text{aa}))$

Because $\mathcal{L}$(aa) has second order variables, our definition of $C(\mathcal{L}(\text{aa}))$ does not guarantee than $C(\mathcal{L}(\text{aa}))$ satisfies Axiom of Choice. It is an open problem whether it does. Therefore we modify the construction. We do not know whether it is a proper modification. Another open problem!

Jensen's J-hierarchy:

Suppose T is a class.

$$\begin{cases} J_0^T & = & \emptyset \\ J_{\alpha+1}^T & = & \text{rud}_T(J_\alpha^T \cup \{J_\alpha^T\}) \\ J_\nu^T & = & \bigcup_{\alpha<\nu} J_\alpha^T, \text{ for } \nu = \cup\nu. \end{cases}$$

Here rud_T includes the operation $x \mapsto x \cap T$.

The definition of $C(\mathrm{aa}) = C(\mathcal{L}(\mathrm{aa}))$

We define the hierarchy (J'_α), $\alpha \in Lim$, as follows:

$$Tr = \{(\alpha, \varphi(\boldsymbol{a})) : (J'_\alpha, \in, Tr \restriction \alpha) \models \varphi(\boldsymbol{a}),$$

$$\varphi(\bar{x}) \in \mathcal{L}(\mathrm{aa}), \boldsymbol{a} \in J'_\alpha, \alpha \in Lim\},$$

where

$$Tr \restriction \alpha = \{(\beta, \psi(\boldsymbol{a})) \in Tr : \beta \in \alpha \cap Lim\},$$

and

$$\begin{array}{lcl} J'_0 & = & \emptyset \\ J'_{\alpha+\omega} & = & \mathrm{rud}_{Tr}(J'_\alpha \cup \{J'_\alpha\}) \\ J'_{\omega\nu} & = & \bigcup_{\alpha<\nu} J'_{\omega\alpha}, \text{ for } \nu \in Lim \\ C(\mathrm{aa}) & = & \bigcup_{\alpha=\cup\alpha} J'_\alpha. \end{array}$$

Stationary tower forcing (see e.g. [8])

- Suppose δ is a Woodin cardinal[2].
- There is a forcing notion $Q = Q_{<\delta}$ such that $|Q| = \delta$ and such that if $G \subseteq Q$ is generic over V then in $V[G]$:
- δ is still a cardinal.
- There is an elementary embedding $j : V \to M$ where M is a transitive class such that $j(\omega_1) = \delta$. and such that $M^\omega \subseteq M$.

[2] A cardinal δ is Woodin, of for all $f : \delta \to \delta$ there is $\kappa < \delta$, closed under f, and $i : V \to M$ with critical point κ such that $V_{j(f(\kappa))} \subseteq M$.

C^* is forcing absolute, assuming PCW

- Suppose $\mathbb{P}$ is a po-set.
- Let G be $\mathbb{P}$-generic.
- Choose a Woodin cardinal $\lambda > |\mathbb{P}|$.
- Let H_1 be generic for the stationary tower forcing $\mathbb{Q}_{<\lambda}$.
- In $V[H_1]$ there is a generic embedding $j_1 : V \to M_1$ such that $V[H_1] \models M_1^\omega \subseteq M_1$ and $j(\omega_1) = \lambda$.
- Hence $(C^*)^{V[H_1]} = (C^*)^{M_1}$ and[3]

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

- Now by elementarity $\mathrm{Th}((C^*)^V) = \mathrm{Th}((C^*_{<\lambda})^V)$.

[3] $C^*_{<\lambda}$ asks whether the cofinality of a linear order is $< \lambda$.

- Since $|\mathbb{P}| < \lambda$, λ is still Woodin in $V[G]$.
- Let H_2 be generic for $\mathbb{Q}_{<\lambda}$ over $V[G]$.
- Let $j_2 : V[G] \to M_2$ be the generic embedding.
- Now $V[G, H_2] \models M_2^\omega \subseteq M_2$ and $j_2(\omega_1) = \lambda$.
- Hence

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[G,H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V,$$

- By elementarity $(C^*)^V \equiv (C^*_{<\lambda})^V \equiv (C^*)^{V[G]}$.

An important property of $C(\mathrm{aa})$: Club Determinacy

- For all α:

$$(J'_\alpha, \in, Tr \restriction \alpha) \models \forall \bar{x}[\mathrm{aa} s \varphi(\bar{x}, \bar{t}, s) \vee \mathrm{aa} s \neg\varphi(\bar{x}, \bar{t}, s)],$$

  where $\varphi(\bar{x}, \bar{t}, s)$ is any formula in $L(\mathrm{aa})$ and $\bar{t}$ is a finite sequence of countable subsets of J'_α.
- CD follows from a proper class of Woodin cardinals [5].
- CD follows from PFA [5].

How does Club Determinacy follow from PCW?

- After some preliminary forcing and absoluteness steps we still have a Woodin cardinal δ and a measurable above.
- Now we use stationary tower forcing and $j : V \to M$.
- We compare[4] $C(\mathrm{aa}_\delta)^V$, $C(\mathrm{aa})^M$, level by level, and show, that they are the same model.
- As we do this, we establish Club Determinacy in V.

[4] aa_δ asks whether there is a club of sets of cardinality $< \delta$ satisfying a formula.

Theorem ([5])

- *Assuming Club Determinacy, every regular $\kappa \geq \aleph_1$ is measurable[5] in $C(\mathtt{aa})$.*
- *Suppose there are a proper class of Woodin cardinals. Then the first order theory of $C(\mathtt{aa})$ is (set) forcing absolute.*

[5] We take, for a big α, all $X \subseteq \kappa$ in J'_α which in J'_α satisfy $\mathtt{aa} s(s \cap \kappa \in X)$.

Proof of the forcing absoluteness of $C(\mathrm{aa})$

- Suppose $\mathbb{P}$ is a forcing notion and δ a Woodin cardinal $> |\mathbb{P}|$.
- Let $j : V \to M$ be the (generic) associated elementary embedding.
- Now $C(\mathrm{aa}) \equiv (C(\mathrm{aa}))^M = C(\mathrm{aa}_\delta)$.
- Let $H \subseteq \mathbb{P}$ be generic over V and $j' : V[H] \to M'$.
- Again: $(C(\mathrm{aa}))^{V[H]} \equiv (C(\mathrm{aa}))^{M'} = (C(\mathrm{aa}_\delta))^{V[H]}$.
- But $(C(\mathrm{aa}_\delta))^{V[H]} = C(\mathrm{aa}_\delta)$, since $|\mathbb{P}| < \delta$

Deeper into $C(\mathrm{aa})$

Deeper understanding of $C(\mathrm{aa})$ requires development of the theory of aa-mice.

Preparation for the definition of aa-premice

- We fix the following notation: $\tau_\xi = \{\mathsf{R}_\in, \mathsf{R}_T, \mathsf{R}_{T^*}\} \cup \{\mathrm{P}_\eta : \eta < \xi\}$, $\tau_\xi^- = \tau_\xi \setminus \{\mathsf{R}_{T^*}\}$.
- Here $\mathsf{R}_\in$ and R_T are binary and $\mathsf{R}_{T^*}, \mathrm{P}_\eta$ $(\eta < \xi)$, are unary.
- We use $(P)_\xi$ to denote a sequence $\langle P_\eta : \eta < \xi \rangle$.

- An *aa-premouse* is a structure $\boldsymbol{J}_\alpha^T = (J_\alpha^T, \in, T, T^*, (P)_\xi)$ in the vocabulary τ_ξ which is a small copy of $C(\mathrm{aa})$.
- Typically, $\boldsymbol{J}_\alpha^T$ is countable, so the semantics of the aa-quantifier makes no sense, and we consider it syntactically only.
- Therefore we deal with aa-theories such as T and T^*.
- T codes the theories of the previous levels.
- T^* is the theory of J_α^T.
- The $(P)_\xi$ is a potentially countable initial segment of a club.

Example

The canonical example of an aa-premouse is

$$\mathcal{N} = (J'_\alpha, \in, Tr \restriction \alpha, Tr_\alpha),$$

where $Tr_\alpha = \{\varphi(\boldsymbol{a}) : (\alpha, \varphi(\boldsymbol{a})) \in Tr$. Note that $\mathcal{N} \in C(\text{aa})$.

Suppose $(J_\alpha^T, \in, T, T^*, (P)_\xi)$ is an aa-premouse. We define

$$\varphi(s, x, \boldsymbol{a}) \sim \varphi'(s, x, \boldsymbol{a}')$$

if and only if

$$\mathrm{aa}\, s(f_{\varphi(s,x,\boldsymbol{a})}(s) = f_{\varphi'(s,x,\boldsymbol{a}')}(s)) \in T^*.$$

The aa-ultrapower of $(J_\alpha^T, \in, T, T^*, (P)_\xi)$, has the set M^* of $\sim$-equivalence classes as its domain. For predicates R we define:

$$R^{M^*}([\varphi_1(s, x, \boldsymbol{a}_1)], \ldots, [\varphi_n(s, x, \boldsymbol{a}_n)]) \iff$$

$$\mathrm{aa}\, s\, R(f_{\varphi_1(s,x,\boldsymbol{a}_1)}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a}_n)}(s)) \in T^*.$$

The *canonical embedding* $j : J'_\alpha \to M^*$ is defined by $j(a) = [x = a]$.

Definition

Let P^* be a new unary predicate symbol and $(P^*)^{M^*} = \{j(a) : a \in J^T_\alpha\}$. We let S^* consist of

$$\psi(P^*, [\varphi_1(s, x, \boldsymbol{a})], \ldots, [\varphi_n(s, x, \boldsymbol{a})]),$$

where $\psi(s, x_1, \ldots, x_n)$ is a τ-formula of $L(\mathrm{aa})$, and

$$\mathrm{aa}\, s\, \psi(s, f_{\varphi_1(s,x,\boldsymbol{a})}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a})}(s)) \in T^*.$$

- We obtain *iterates* $(M_\beta, E_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta)$ of the aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$.
- An aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$ is an **aa-mouse** if its β'th iterate $(M_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta)$ is well-founded for all $\beta < \omega_1$.
- In this case we say that the aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$ is *iterable*.

Proposition

Let $\langle (M_\beta, E_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta), j_{\beta,\gamma} : \beta \leq \gamma \leq \omega_1 \rangle$ *be an aa-iteration of aa-mice. Then for all formulas* $\varphi(\boldsymbol{a})$ *of stationary logic in vocabulary* $\tau^-_{\omega_1}$ *and all* $\boldsymbol{a} \in M_{\omega_1}$*:*

$$\varphi(\boldsymbol{a}) \in T^*_{\omega_1} \iff (M_{\omega_1}, E_{\omega_1}, T_{\omega_1}, (P^{\omega_1})_{\omega_1}) \models \varphi(\boldsymbol{a}).$$

Lemma

Suppose

$$(M_0, \in, T_0, T_0^*, (P)_0) \prec (J'_{\omega\alpha}, \in, Tr\restriction\omega\alpha, Tr_{\omega\alpha}, (P')_0),$$

where α is a limit ordinal and M_0 is countable. Then M_{ω_1} does not have new reals over those in M_0.

Theorem ([5])

If Club Determinacy holds, then CH holds in $C(\mathrm{aa})$.

Theorem ([5])

If club determinacy holds, there is a Δ^1_3 well-ordering of the reals in $C(\mathrm{aa})$. The reals form a countable Σ^1_3-set.

Proof.

The canonical well-order $\prec$ of $C(\mathrm{aa})$ satisfies:

$$x \prec y \iff \exists z \subseteq \omega(\ z \text{ codes an aa-mouse } M \text{ such that}$$

$$x, y \in M \text{ and } M \models \text{“}x \prec y\text{”}).$$

The right hand side of the equivalence is Σ^1_3 and the claim follows. □

A summary of this lecture

- Stationary logic has a completeness theorem.
- Stationary logic gives rise to $C(\mathrm{aa})$.
- Assuming large cardinals, $C(\mathrm{aa})$ is forcing absolute, has measurable cardinals, and satisfies CH.

The next Lecture:

We learn:

- Why the second order constructible universe $C(\mathcal{L}^2)$ is the strongest?
- How to force the powerful Henkin quantifier constructible universe part ways with $C(\mathcal{L}^2)$?
- Why quantifying over countable sets gives a perfect (?) inner model?

Tutorial Lecture Three

We learn:

- Why the second order constructible universe $C(\mathcal{L}^2)$ is the strongest?
- How to force the powerful Henkin quantifier constructible universe part ways with $C(\mathcal{L}^2)$?
- Why quantifying over countable sets gives a perfect (?) inner model?

Gödel introduced HOD in 1946 [4].

Theorem ([11])

$C(\mathcal{L}^2) = \mathrm{HOD}$.

$C(\mathcal{L}^2) \subseteq \mathrm{HOD}$

- Let (L'_α) be the hierarchy building $C(\mathcal{L}^2)$.
- We use induction on α to show $L'_\alpha \in \mathrm{HOD}$.
- Suppose $X \in L'_{\alpha+1}$, i.e. $X = \{a \in L'_\alpha : L'_\alpha \models \varphi(a, \vec{b})\}$, where $\varphi(x, \vec{y})$ is a second order formula in the language of set theory and $\vec{b} \in L'_\alpha$.
- Now $X = \{a : a \in L'_\alpha \wedge L'_\alpha \models \varphi(a, \vec{b})\}$, whence $X \in \mathrm{HOD}$ as the truth predicate "$\models$" of second order logic is a set-theoretical predicate.

$C(\mathcal{L}^2) \supseteq \mathrm{HOD}$

- Let $X \in \mathrm{HOD}$.
- There is a first order $\varphi(x, \vec{y})$ and ordinals $\vec{\beta}$ such that for all a

$$a \in X \iff \varphi(a, \vec{\beta}).$$

- By Levy Reflection there is an α such that $X \subseteq V_\alpha$ and for all $a \in V_\alpha$

$$a \in X \iff V_\alpha \models \varphi(a, \vec{\beta}).$$

- Since we proceed by $\in$-induction, we may assume $X \subseteq C(\mathcal{L}^2)$.
- Let γ be big enough that $X \subseteq L'_\gamma$ and $|L'_\gamma| \geq |V_\alpha|$.
- We show now that $X \in L'_{\gamma+1}$ by giving a second order formula $\Phi(x, y, \vec{z})$ such that

$$X = \{a \in L'_\gamma : L'_\gamma \models \Phi(a, \alpha, \vec{\beta})\}.$$

- We know

$$X = \{a \in L'_\gamma : V_\alpha \models \varphi(a, \vec{\beta})\}.$$

- Intuitively, X is the set of $a \in L'_\gamma$ such that on L'_γ some $(M, E, P, a^*, \vec{\beta^*}) \cong (V_\alpha, \in, \alpha, a, \vec{\beta})$ satisfies $\varphi(a^*, \vec{\beta^*})$.

- We formalize the intuitive idea:
- Let $\theta(x, \vec{z})$ be a second order formula of the vocabulary $\{E, P\}$, E binary ("$\in$"), P unary ("the class of all ordinals"), such that:
- $(M, E, P) \models \theta(a^*, \vec{\beta^*}) \iff$ there is

$$\pi : (M, E, P) \cong (V_\delta, \in, \delta)$$

such that

$$(V_\delta, \in) \models \varphi(\pi(a^*), \pi(\vec{\beta^*})).$$

Assume for a moment we have such a θ.

The following are equivalent, showing that $X \in L'_{\gamma+1}$:

(1) $a \in X$

(2) $L'_\gamma \models \exists M \exists E((M, E, \alpha) \models \theta(a, \vec{\beta}) \wedge \mathsf{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M)$.

(1) $a \in X$

(2) $L'_\gamma \models \exists M \exists E((M, E, \alpha) \models \theta(a, \vec{\beta}) \wedge \mathrm{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M)$.

- (1) $\rightarrow$ (2) : Suppose $a \in X$. Thus $V_\alpha \models \varphi(a, \vec{\beta})$.
- Let $M \subseteq L'_\gamma$ and $E \subseteq M^2$ s. t. $\alpha + 1, \mathrm{TC}(a), \vec{\beta} \in M$ and there is an isomorphism $f : (V_\alpha, \in, \alpha, a, \vec{\beta}) \cong (M, E, P, a^*, \vec{\beta^*})$. Remember $|L'_\gamma| \geq |V_\alpha|$.
- We can assume $P = \alpha$, $a^* = a$ and $\vec{\beta^*} = \vec{\beta}$ by doing a partial Mostowski collapse[6] for (M, E), since $\alpha + 1, \mathrm{TC}(a), \vec{\beta} \in M$.
- Then $(M, E, \alpha) \models \varphi(a, \vec{\beta})$, whence $(M, E, \alpha) \models \theta(a, \vec{\beta})$, i.e. (2).

[6] In the Mostowski collapse everything that is transitive collapses onto itself.

(1) $a \in X$

(2) $L'_\gamma \models \exists M \exists E((M, E, \alpha) \models \theta(a, \vec{\beta}) \wedge \mathrm{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M)$.

- (2) $\to$ (1) : Suppose $M \subseteq L'_\gamma$, $P \subseteq M$, and $E \subseteq M^2$ such that $\mathrm{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M$ and $(M, E, P) \models \theta(a, \vec{\beta})$.
- We may assume $E \restriction \mathrm{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} = \in \restriction \mathrm{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\}$.
- There is an isomorphism $\pi : (M, E, P) \cong (V_\alpha, \in, \alpha)$ such that $(V_\alpha, \in) \models \varphi(\pi(a), \pi(\vec{\beta}))$.
- But $\pi(a) = a$ and $\pi(\vec{\beta}) = \vec{\beta}$.
- So in the end $(V_\alpha, \in) \models \varphi(a, \vec{\beta})$. We have proved (1).

Construction of the formula θ

$\theta(x, z)$ is the conjunction, in the vocabulary $\{E, P\}$, of:

1. E is extensional and well-founded (i.e. there is no infinite sequence (a_n) such that $a_{n+1} E a_n$ for all n.)
2. The class of the ordinals of the model is P.
3. If the cumulative hierarchy is built inside M using E as the ε-relation, then at successor stages M_{a+1} *every* subset of M_a is in M_{a+1}.
4. $\varphi(x, z)$ translated into the $\{E\}$-vocabulary.

- Define $\mathrm{HOD}_1 = C(\mathcal{L}(H))$, where H is the Henkin quantifier

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x,y,u,v) \iff$$

$$\exists f, g \forall x, u \varphi(x, f(x), u, g(u)) \iff {}^{7}$$

$$\forall x \forall u \exists y \exists v (=(x,y) \wedge =(u,v) \wedge \varphi(x,y,u,v))$$

- Equivalently, $\mathrm{HOD}_1 = C(\Sigma^1_1)$.

[7] [14]

Note that for all β and $A \in \mathrm{HOD}_1$:

- $\{\alpha < \beta : \mathrm{cf}^V(\alpha) = \omega\} \in \mathrm{HOD}_1$
- $\{(a, b) \in A^2 : |a|^V \leq |b|^V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : \alpha \text{ cardinal in } V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : (2^{|\alpha|})^V = (|\alpha|^+)^V\} \in \mathrm{HOD}_1$

Lemma ([6])

1. $C^* \subseteq \mathrm{HOD}_1$.
2. $C(Q_1^{MM,<\omega}) \subseteq \mathrm{HOD}_1$.
3. *If* $0^\sharp$ *exists, then* $0^\sharp \in \mathrm{HOD}_1$ *(by 1).*

- Naturally, $\mathrm{HOD}_1 = \mathrm{HOD}$ is consistent, since we can assume $V = L$.
- Also $\mathrm{HOD}_1 \neq \mathrm{HOD}$ is consistent relative to Con(ZF) ([6]).
- Now we ask the same questions in the presence of large cardinals.

Theorem ([6, 9])

It is consistent, relative to the consistency of a supercompact cardinal that $\mathrm{HOD} = \mathrm{HOD}_1$ *and there is also a supercompact cardinal.*

We use [10] where it is proved that $V = \mathrm{HOD}$ is consistent with a supercompact. The coding used in [10] gives also that $V = \mathrm{HOD}_1$ is consistent with a supercompact.

Theorem ([6, 9])

It is consistent, relative to the consistency of a supercompact cardinal that for some λ:

$$\{\kappa < \lambda : \kappa \text{ weakly compact}\} \notin \mathrm{HOD}_1,$$

and there is a supercompact cardinal.

Corollary

$\mathrm{HOD} \neq \mathrm{HOD}_1$ *is consistent with a supercompact.*

- [10] shows that one can start with any model with a supercompact cardinal and force $V = \mathrm{HOD}$ preserving the supercompact cardinal.
- It is enough to code subsets of cardinals κ such that $\kappa = \beth_\kappa$.
- Menas codes subsets into the relation $2^{\aleph_{\alpha+1}} = \aleph_{\alpha+\omega+3}$.
- A set $X \subseteq \aleph_\nu$ ends up being definable as

$$\{\alpha < \aleph_\nu : 2^{\aleph_{e(\nu)+\alpha+1}} = \aleph_{e(\nu)+\alpha+3}\},$$

  where $e(\nu)$ is the νth ordinal α such that $\alpha = \beth_\alpha$.
- Hence his model satisfies $V = \mathrm{HOD}_1$.

To be done on the blackboard

- We assume V_0 has a supercompact cardinal and satisfies $V = \mathrm{HOD}_1$.
- Let $\kappa_n, n < \omega$ be a sequence of weakly compact cardinals and $\lambda = \sup_n \kappa_n$.
- Let $\mathbb{D}_\delta$ be the forcing notion for adding a Cohen subset of the regular cardinal δ.
- We proceed as in [7]. Let $\eta < \kappa$ be two regular cardinals. We denote by $\mathbb{R}_{\eta,\kappa}$ the Easton support iteration of $\mathbb{D}_\delta$ for $\eta \leq \delta \leq \kappa$.

To be done on the blackboard

- The forcing $\mathbb{R}_{\kappa^+_{n-1},\kappa_n}$, where for $n = 0$ we take $\kappa_{-1} = \omega_1$, we denote by $\mathbb{P}_n$.
- Note that forcing with $\mathbb{P}_n$ preserves the weak compactness of κ_n.
- Note that $\mathbb{P}_n * \mathbb{D}_{\kappa_n}$ is forcing equivalent to $\mathbb{P}_n$.
- Let $\mathbb{Q}$ be the full support product of $\mathbb{P}_n, n < \omega$. Let $V^* = V_0^{\mathbb{Q}}$.
- $\mathbb{Q}$ can be decomposed as $\mathbb{Q}_n \times \mathbb{P}_n \times \mathbb{Q}^n$ where $\mathbb{Q}_n$ has cardinality κ_{n-1} and $\mathbb{Q}^n$ is κ_n^+ closed.
- Hence $\mathbb{Q}_n$ and $\mathbb{Q}^n$ do not change the weak compactness of κ_n, which is preserved by $\mathbb{P}_n$.
- For $n < \omega$, the cardinal κ_n is weakly compact in V^*.

To be done on the blackboard

- As in [7], we define in $V^{\mathbb{P}_n}$ a forcing $\mathbb{S}_n$ to be the canonical forcing which introduces a κ_n homogeneous Soulin tree. It kills the weak compactness of κ_n.
- Let $\mathbb{T}_n$ be the forcing which introduces a branch through the tree forced by $\mathbb{S}_n$.
- $\mathbb{S}_n * \mathbb{T}_n$ is forcing equivalent to $\mathbb{D}_{\kappa_n}$.
- If we force with $\mathbb{T}_n$ over $V^{\mathbb{P}_n * \mathbb{S}_n}$, we regain the weak compactness of κ_n.
- Also a generic object for $\mathbb{D}_{\kappa_n}$ introduces a generic object for $\mathbb{T}_n$.

To be done on the blackboard

- Let $V_1^* = V_0^{\mathbb{Q}}$.
- Let G_n be the generic filter in $\mathbb{P}_n$, introduced by $\mathbb{Q}$.
- The model V_3^* is the model one gets from V_1^* by forcing over it with the full support product of $\mathbb{D}_{\kappa_n}$. ($\mathbb{D}_{\kappa_n}$ is as realized according to G_n.).
- Let $H_n \subseteq \mathbb{D}_{\kappa_n}$ be the generic filter introduced by this forcing.
- Note that V_3^* can also be obtained from V_0 by forcing with $\mathbb{Q}$. In particular both in V_1 and in V_3^* the cardinals κ_n are weakly compact for every $n < \omega$. Let V_3 be an extension of V_3^* by adding a Cohen real $a \subseteq \omega$.
- Let $\mathbb{A}$ be the Cohen forcing on ω.
- Let $V_1 = V_1^*(a)$. Both V_1 and V_3 are obtained by forcing over V_0 with $\mathbb{Q} \times \mathbb{A}$ which is a homogenous forcing notion. Hence $HOD^{V_1} = HOD^{V_3}$.
- Again we did not kill the weak compactness of the cardinals κ_n.

To be done on the blackboard

- Each H_n introduces a generic filter for the forcing $\mathbb{S}_n$ (As defined according to G_n). Let $K_n \subseteq \mathbb{S}_n$ be this generic filter.
- We define $V_2 = V_1[a, \langle K_n | n \notin a\rangle, \langle H_n | n \in a\rangle]$.
- For $n < \omega$ we define $W_n = V_0(a, \langle G_i | i \leq n\rangle, \langle K_i | i \leq n, i \notin a\rangle, \langle H_i | i \leq n, i \in a\rangle)$.
- If $n \in a$ then W_n is obtained from V_0 by a product of $\mathbb{P}_n * \mathbb{D}_n$ and some forcings of size $< \kappa_n$. Since $\mathbb{P}_n * \mathbb{D}_n$ preserves the weak compactness of κ_n, κ_n is weakly compact in W_n.
- If $n \notin a$ then K_n generates a tree on κ_n which is still Souslin in W_n. (Small forcings do not change the Souslinity of a tree.) So κ_n is not weakly compact in W_n.
- κ_n is weakly compact in W_n iff $n \in a$.

To be done on the blackboard

- By the standard arguments analysing the power-set of a cardinal δ under a forcing which is the product of a forcing of size $\mu < \delta$, a forcing of size δ which is μ^+-distributive, and a forcing which is δ^+-distributive:
- For $n < \omega$ $P(\kappa_n)^{V_2} = P(\kappa_n)^{W_n}$.
- $V_2 \models a = \{n < \omega | \kappa_n \quad \text{is weakly compact}\}$.
- $a \in HOD^{V_2}$.

To be done on the blackboard

- The proof of the Theorem will be finished if we show that $HOD_1^{V_2} = V_0$.
- For an ordinal α let $L^1_\alpha, L^2_\alpha, L^3_\alpha$ be the α-th step of the construction of $(C(\Sigma^1_1))^{V_1}$, $(C(\Sigma^1_1))^{V_2}$, $(C(\Sigma^1_1))^{V_3}$ respectively.
- $(\star)$ For every α: $L^1_\alpha = L^2_\alpha = L^3_\alpha$.
- The proof of $(\star)$ is by induction on α where the cases $\alpha = 0$ and α limit are obvious.
- By the induction assumption on α we can put $M = L^1_\alpha = L^2_\alpha = L^3_\alpha$.
- Note that $M \in V_0$ since $M \in HOD^{V_1} = V_0$.
- Let $\Phi(\vec{x})$ be a Σ^1_1 formula and let $\vec{b}$ be a vector of elements of M.

To be done on the blackboard

Lemma

The following are equivalent

1. $(M \models \Phi(\vec{b}))^{V_1}$.
2. $(M \models \Phi(\vec{b}))^{V_3}$.
3. $(M \models \Phi(\vec{b}))^{V_2}$.

To be done on the blackboard

- Without loss of generality, $\Phi(\vec{x})$ has the form $\exists X \Psi(X, \vec{x})$.
- Both V_1 and V_3 are obtained form V_0 by forcing over V_0 with $\mathbb{Q} \times \mathbb{A}$. This forcing is homogeneous.
- M and all the elements of the vector $\vec{b}$ are in V_0.
- So (1) is clearly equivalent to (2).
- Suppose that $(M \models \Phi(\vec{b}))^{V_2}$. Let $Z \subseteq M$ be the witness for the existential quantifier of Φ. Then $(M \models \Psi(Z, \vec{b}))^{V_2}$. But all the quantifiers of Ψ are first order, so $(M \models \Psi(Z, \vec{b}))^{V_3}$.
- So (3) implies (2), and hence (1).
- For the other direction, if $(M \models \Phi(\vec{b}))^{V_3}$, then we know that $(M \models \Phi(\vec{b}))^{V_1}$. Let $Z \in V_1$ satisfy $M \models \Psi(Z, \vec{b})$. So $(M \models \Psi(Z, \vec{b}))^{V_2}$, and therefore $(M \models \Phi(\vec{b}))^{V_2}$.

To be done on the blackboard

- It follows from the lemma that every Σ^1_1 formula defines the same subset of M in V_1, V_2 and V_3.
- It follows that $L^1_{\alpha+1} = L^2_{\alpha+1} = L^3_{\alpha+1}$.
- This proves the lemma and the theorem.
- Conclusion: Large cardinals cannot decide the question "$\mathrm{HOD} = \mathrm{HOD}_1$?"

Quantifying over countable sets[8]: $C^2(\omega)$ [9]

- $\mathcal{L}^2(\omega)$ is second order logic where the second order variables range over countable sets.
- $C^2(\omega) =_{\text{def}} C(\mathcal{L}^2(\omega))$.
- $C^* \subseteq C^2(\omega)$.
- Consistently $C^2(\omega) \not\subseteq C^*$: Force over L a Δ^1_3-non constructible real. That real is in $C^2(\omega)$, but the forcing is CCC, so $C(\mathrm{aa}) = C^* = L$. Likewise $C^2(\omega) \not\subseteq C(\mathrm{aa})$.
- Consistently $C(\mathrm{aa}) \not\subseteq C^2(\omega)$: Start with L. Add a Cohen real. Still $C^2(\omega) = L$ as the forcing is homogeneous. Now code by further forcing the Cohen real into stationarity of some L-stationary sets. The forcing does not add new countable sets, so still $C^2(\omega) = L$. But now the Cohen real is in $C(\mathrm{aa})$.

[8] Joint work with Menachem Magidor.

Quantifying over countable sets[9]: $C^2(\omega)$

- $ZFC \vdash C^2(\omega) \subseteq HOD^{C_{\omega_1\omega_1}}$. Hence $Th(C^2(\omega))$ is forcing absolute, assuming a proper class of Woodin limits of Woodins. (Woodin)
- $V = C^2(\omega)$ implies there are no measurable cardinals.
- ω_1^V is strongly Mahlo in $C^2(\omega)$, assuming a Woodin limit of Woodins. [9]
- $C^2(\omega)$ contains, for every n, the inner model with n Woodin cardinals.
- If there is a proper class of Woodin limits of Woodin cardinals, then every regular cardinal is measurable in $C^2(\omega, \mathrm{aa})$ and $C^2(\omega, \mathrm{aa})$ is forcing absolute [9].
- CH?

[9] Joint work with Menachem Magidor.

Recent developments

1. Assume a proper class of Woodin cardinals. The reals of $C(\mathrm{aa})$ (and also of C^*) are in M_1. $C(\mathrm{aa})$ has no inner model with a Woodin cardinal. (Magidor-Schindler)
2. Assume Club Determinacy. Then Ultrapower Axiom and GCH hold in $C(\mathrm{aa})$. (Goldberg-Steel)
3. Assume Club Determinacy. If κ is regular in $C(\mathrm{aa})$ with $cof(\kappa) \geq \omega_2^V$, then $o(\kappa)^{C(\mathrm{aa})} \geq 2$. Moreover, then $o(\omega_3^V)^{C(\mathrm{aa})} \geq 3$. (Goldberg-Rajala)
4. If $V = L^\mu$, then $V = C(\mathrm{aa})$. (Goldberg, Magidor, Schindler, Steel)
5. There is ongoing investigation on what kind of mice can be found inside models such as C^* and $C(\mathrm{aa})$. (Goldberg, Magidor, Schindler, Steel)

A summary of this lecture

- The second logic version of L is actually HOD.
- The Henkin quantifier version of L, called HOD_1, can be smaller than HOD
- Quantifying over countable subsets yields an inner model $C^2(\omega, \mathrm{aa})$ which is, assuming large cardinals, forcing absolute and has measurable cardinals. Does it have CH?

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Thank you!