Equidecomposition and discrepancy: Part 3

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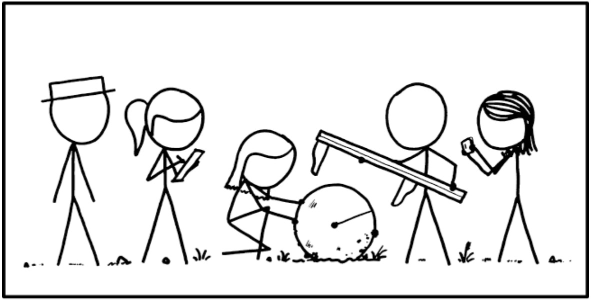
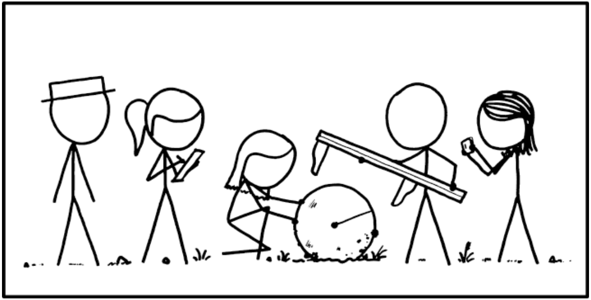


MATH BREAKTHROUGH: DIMENSIONAL ANALYSTS HAVE DISCOVERED A REAL UNIT CIRCLE. ONCE THEY MEASURE IT, UNITS CAN FINALLY BE ADDED TO ALL OUR GEOMETRY TEXTBOOKS.



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Alt text: They're continuing to search for a square with the same area as the circle, as efforts to construct one have run into difficulties.

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Today we will show instances of how to bound $D(\mu)$ for different choices of actions and μ.

The golden ratio

Consider the following sequence of continued fractions.

$$1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}$$

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Note that $L = 1/\phi$ where ϕ is the golden ratio.

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We can actually show that this converges very quickly.

$$\left|\frac{1}{\phi} - \frac{F_{n-1}}{F_n}\right| \leq \frac{1}{F_{n+1}F_n}$$

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So for $[a, b) \subseteq \mathbb{T}$, it follows that

$$|(b-a) - |\{1/\phi, \ldots, F_n/\phi\}|/F_n| \leq 2/F_n$$

Estimates where F_N is replaced by an arbitrary natural number are also possible.

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These estimates are not enough to get the desired convergence of the series of averages.

The Erdös-Turán inequality

Theorem (Erdös-Turán)

There are constants C_1, C_2 such that for all finitely supported probability measures μ on $\mathbb{T}$ and all $m \in \mathbb{N}$,

$$D(\mu) \leq C_1 \frac{1}{m+1} + C_2 \sum_{h=1}^{m} \left| \frac{\hat{\mu}(h)}{h} \right|$$

where $\hat{\mu}$ is the Fourier transform of μ.

Some Fourier analysis

The Fejér kernel

$$F_m(x) = \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} e(kx)$$

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Then we consider convolutions $|(f * F_m)(s)|$, which for a good value of s is bounded between $\sup_x f(x) - \frac{C_1}{m}$ and $C_2 \sum_{h=1}^{m} \frac{\hat{\mu}(h)}{h}$.

A “basic” calculation using Erdös-Turan

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The inner sum has the form of a geometric series.

Continued calculation

Isolating this geometric series we have:

$$\left|\sum_{n=0}^{N-1} e(nz)\right| = \left|\frac{e(Nz)-1}{e(z)-1}\right| = \left|\frac{e(Nz/2)(e(Nz/2)-e(-Nz/2))}{e(z/2)(e(z/2)-e(-z/2))}\right|$$

$$= \left|\frac{e(Nz/2)\sin(\pi Nz)}{e(z/2)\sin(\pi z)}\right| = \left|\frac{\sin(\pi Nz)}{\sin(\pi z)}\right| \leq \frac{1}{|\sin(\pi z)|} \leq \frac{1}{2\langle z\rangle}$$

where $\langle z\rangle$ is the distance to the closest integer.

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A similar fact is true for almost every $u \in \mathbb{T}$.

Since we only used a single irrational, the bounds that we get on $D(\mu)$ are still inadequate to get convergence of the averaging procedure using uniform measures over finite sets.

A lemma for almost every u

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For every $\epsilon > 0$, for almost every $u \in \mathbb{T}$, there are finitely many $h > 0$ such that $\langle hu \rangle < h^{-1-\epsilon}$.

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For each $h > 0$, Let E_h be the set of $u \in \mathbb{T}$ for which we have the condition in the lemma. Clearly, u is in E_h if and only if for some $m \leq h$, u lies in the interval of length $2h^{-2-\epsilon}$ around m/h.

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Summing over m, we have that the measure of E_h is at most $2(h+1)/h^{-2-\epsilon}$.

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Summing over m, we have that the measure of E_h is at most $2(h+1)/h^{-2-\epsilon}$.

Note that these measures are summable, so by the Borel-Cantelli lemma the set of u that lie in infinitely many E_h has measure 0. The complement is the desired set.

More irrationals

Option 1:

- We can choose $u_1, \ldots u_d$ to be either random or algebraic irrationals in $\mathbb{T}$ and repeat the argument above using Erdös-Turan with a uniform measure. For algebraic irrationals this requires a (very difficult) theorem of Schmidt.

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- To pass higher dimensions we can then consider product actions, where $\mathbb{Z}^{dk}$ acts on $\mathbb{T}^k$.

Option 2:
We can choose $u_1, \dots u_d \in \mathbb{T}^k$ randomly and use a higher dimensional version of the Erdös-Turan inequality due to Koksma.

About averaging with random walks

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The expected value $\widehat{\rho^{*k}}(h) \approx k^{-d/2}$.

Reducing the number of pieces

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- The number of pieces depends on the bound on the flow across an edge, which is the convergent infinite sum from the theorem that circle squaring is possible with algebraic irrational coordinates.
- We use a computer to explicitly compute initial terms of the discrepancy in order to improve the bound.

Open questions

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2. Are the disk and square equidecomposible using F_σ sets?