Equidecomposition and discrepancy: Part 2

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Recap

Let a be the action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ given by translations $u_1, \dots u_d$ that we will choose later.

We are interested in flows of functions f with $\int f d\lambda = 0$. We're particularly interested in $1_A - 1_B$ where A, B are the disk and the square.

Finding flows amounts to proving numerical bounds on functions of the form $|Av_\rho(f)(x)| = |\int f d\rho^x|$.

Assuming nice properties of f, for any measure μ, $|\int f d\mu|$ can be bounded in terms of $D(\mu) = \sup_I |\lambda(I) - \mu(I)|$.

Bounding $D(\mu)$ depends in a fundamental way on the choice of the action.

A first theorem

Laczkovich's 1990 solution to Tarski's circle squaring problem used translations chosen randomly. In this paper, he asked if the same was possible with algebraic irrational translations.

Theorem (Marks-U)

Known circle squaring results are possible using translations with algebraic irrational coordinates.

Discrepancy of functions

For a probability measure μ on $\mathbb{T}^k$ and a function $f : \mathbb{T}^k \to \mathbb{R}$, let $D(f, \mu) = |\int f d\mu - \int f d\lambda|$ where λ is Lebesgue measure.

Note that when $\int f d\lambda = 0$, the values $|Av_\rho(f)(x)|$ are of the form $D(f, \rho^x)$ where $\rho^x(\gamma \cdot x) = \rho(\gamma)$ for $\gamma \in \mathbb{Z}^d$ and 0 otherwise.

Discrepancy of measures

For a probability measure μ, let $D(\mu) = \sup_I |\mu(I) - \lambda(I)|$ where the supremum is over boxes $I \subseteq \mathbb{T}^k$.

When f is the characteristic function of a set A, Laczkovich showed that the dimension of the boundary of A, $\mathrm{Dim}(\partial A)$, can be used to bound $D(1_A, \mu)$ in terms of $D(\mu)$.

Riemann integration

For a bounded function f on $\mathbb{T}^k$, let
$\tau(f,\delta) = \int \sup_{y\in B_\delta(x)} f(y) - \inf_{y\in B_\delta(x)} f(y)dx$

Fact
$f : \mathbb{T}^k \to \mathbb{R}$ is Riemann integrable if and only if $\lim_{\delta\to 0^+} \tau(f,\delta) = 0$.

We will be interested in functions with explicit bounds on $\tau(f,\delta)$. For characteristic functions this is equivalent to the set having small boundary. Note also that Holder functions also have bounds on the integrand of $\tau(f,\delta)$ and hence on $\tau(f,\delta)$.

A second theorem

For a finitely supported probability measure ρ on $\mathbb{Z}^d$, let *rank*(ρ) be the rank of the subgroup of $\mathbb{Z}^d$ generated by the support of ρ.

Theorem (Bernshteyn-Tserunyan-U)

For almost every $\vec{u} = u_1, \ldots u_d \in \mathbb{T}^k$, if f, g are functions with $\int f d\lambda = \int g d\lambda$ and $\tau(f, \delta), \tau(g, \delta)$ are $O(\delta^\epsilon)$ and ρ is a probability measure with $rank(\rho) > 2k/\epsilon$, then there are functions $\bar{f}$ and f_γ for $\gamma \in \mathrm{supp}(\rho)$ such that

1. $f = \bar{f} + \sum_{\gamma \in \mathrm{supp}(\rho)} f_\gamma$
2. $g = \bar{f} + \sum_{\gamma \in \mathrm{supp}(\rho)} \rho(\gamma)(\gamma \cdot_{\vec{u}} f)$.

Bounding $D(f, \mu)$ in terms of $D(\mu)$

Theorem (Bernshteyn-Tserunyan-U)

If $\tau(f, \delta)$ is $O(\delta^\alpha)$, then there is a constant C such that for all probability measures μ, $D(f, \mu) \leq CD(\mu)^{\alpha/k}$.

A sketch of the proof

Let $\mathcal{P}_N$ be the natural partition of $\mathbb{T}^k$ into half-open boxes of side length $1/N$. Define $H_0(x) = \inf_{y \in P} f(y)$ where $P \in \mathcal{P}_N$ is unique with $x \in P$. And define H_1 similarly with sup.

Then we have $H_0 \leq f \leq H_1$.
It follows that

$$\Big(\int H_0 d\mu - \int H_0 d\lambda \Big) + \Big(\int H_0 d\lambda - \int f d\lambda \Big) \leq \int f d\mu - \int f d\lambda$$
$$\leq \Big(\int H_1 d\mu - \int H_1 d\lambda \Big) + \Big(\int H_1 d\lambda - \int f d\lambda \Big).$$

Taking the absolute value and rearranging with the triangle inequality we have

$$D(f, \mu) \leq \max_{i=0,1} D(H_i, \mu) + \max_{i=0,1} \Big| \int H_i d\lambda - \int f d\lambda \Big|.$$

Bounding $\left| \int H_i d\lambda - \int f d\lambda \right|$

We have

$$\left| \int H_i d\lambda - \int f d\lambda \right| \leq \tau(f, \sqrt{k}/N)$$

since for all x, the ball of radius $\sqrt{k}/N$ contains unique $P \in \mathcal{P}_N$ with $x \in P$.

Having $N \approx 1/D(\mu)^{1/k}$ gives the correct bound.

Bounding $D(H_i, \mu)$

This involves some "harmonic analysis". For simplicity we focus on the $k = 1$ case. The goal is to write H_i efficiently as a sum of characteristic functions of intervals.

At the end of the argument we will need to take $N \approx 1/D(\mu)$, so decomposing H_i as the trivial sum of N characteristic functions is unhelpful.

Consider the following recursive procedure: At step $n \geq 0$, define $\alpha_P = \int_P H_i dx$ for $P \in \mathcal{P}_{2^n}$ and redefine H_i to be $H_i - \sum_{P \in \mathcal{P}_n} \alpha_P 1_P$.

At the end, the original function H_i is written as $\sum_P \alpha_P 1_P$.

It follows that $D(H_i, \mu) \leq \sum_P D(\alpha_P 1_P, \mu) \leq D(\mu) \sum_P |\alpha_P|$.

Bounding $D(H_i, \mu)$ continued

Note that for a step function H defined on $\mathcal{P}_N$ and $P \in \mathcal{P}_{2^n}$ for some n, the quantity

$$\int_P \left| H - \frac{1}{\lambda(P)} \int_P H \right|$$

is at most

$$TV_P(H)\lambda(P)$$

Note that each $|\alpha_P|$ is at most a term of the above form and following the inductive procedure above we get that $\sum_P |\alpha_P| \leq 4TV(H_i)$.

Finally, $TV(H_i)$ can be bounded by $2N\tau(f, 2/N)$ using the fact that $\tau(f, 2/N)$ "detects" the sup/inf used in defining H_i.

Combining our estimates we have

$$D(H_i, \mu) \leq 4D(\mu)TV(H_i) \leq 4D(\mu)N\tau(f, 2/N) \leq 4\tau(f, 2/N)$$

Product measures

Proposition

Suppose that $\mu = \prod_{i=1}^{k} \mu_i$ where each μ_i is a measure on $\mathbb{T}$. Then we have $D(\mu) \leq 2^{k-1} \max_i D(\mu_i)$.

We prove the case $k = 2$. Fix an interval $I = I_1 \times I_2$.

We compute

$$
\begin{aligned}
|\mu(I) - \lambda(I)| =& |\mu_1(I_1)\mu_2(I_2) - \bar{\lambda}(I_1)\bar{\lambda}(I_2)| \\
=& |\mu_1(I_1)\mu_2(I_2) - \bar{\lambda}(I_1)\mu_2(I_2) + \bar{\lambda}(I_1)\mu_2(I_2) - \bar{\lambda}(I_1)\bar{\lambda}(I_2)| \\
\leq& \mu_2(I_2)|\mu_1(I_1) - \bar{\lambda}(I_1)| + \bar{\lambda}(I_1)|\mu_2(I_2) - \bar{\lambda}(I_2)| \\
\leq& 2 \max_{i=1,2} D(I_i, \mu_i)
\end{aligned}
$$

Then take the supremum over I_1, I_2.

This suggests the following:

Lemma (Marks-U)

For $\mu = \prod_{i=1}^{k} \mu_i$ where μ_i is a uniform measure over a finite subset of $\mathbb{T}$ and a set A with $\tau(1_A, \delta)$ is $O(\delta^\alpha)$, there is a constant such that $D(1_A, \mu) \leq C(\max_i D(\mu_i))^\alpha$.

Note that we've dropped the $1/k$ from the exponent in the previous theorem.

A basic lemma about discrepancy

For a finite set F, we write μ_F for the uniform probability measure on F.

Lemma

Suppose $F = \{x_0, \ldots, x_{n-1}\} \subseteq [0, 1)$ *is a finite set where* $x_0 < \ldots < x_{n-1}$. *Then for all* $i < n$, $|x_i - \frac{i}{n}| \leq D(\mu_F)$.

Note that $|x_i - \frac{i}{n}| = D([0, x_i), \mu_F) \leq D(\mu_F)$.