Equidecomposition and discrepancy

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Three parts:

1. Equidecomposition, flows and averaging.
2. Main theorems. Bounding discrepancy of functions.
3. Bounding discrepancy of measures.

These talks contain joint work with Andrew Marks and with Anton Bernshteyn and Anush Tserunyan.

The Banach Tarski paradox

Theorem (Banach-Tarski(AC))

The unit ball in $\mathbb{R}^3$ can be partitioned into 5 pieces which can be moved by isometries to partition two unit balls.

We say that the unit ball is *paradoxical*.

Part of a larger project in the early 20th century to understand

1. the foundations and limitation of measure theory,
2. how measure theory is related to classical ideas such as decomposing polygons into congruent sets and
3. the role of the axiom of choice in the above.

Questions

Question (Borel Ruziewicz problem)

For $n \geq 2$, is Lebesgue measure the unique finitely additive, isometry invariant measure defined on the Borel sets?

This question is open, but has a positive answer when Borel sets are replaced by Lebesgue measurable sets. This is due to Drinfeld, Margulis and Sullivan.

Question (Tarski's circle squaring problem)

Given a disk and a square in the plane with the same area, are they equidecomposible using isometries?

This was solved positively by Laczkovich in 1990 with more recent work by Grabowski-Máthé-Pikhurko, Marks-U and Máthé-Noel-Pikhurko.

Equidecomposition

We start by defining a natural real valued notion of equidecomposition.

Definition

Functions $f, g : X \to \mathbb{R}$ are *equidecomposible* using transformations $T_1, \ldots, T_n$ if there are functions $f_0, f_1, \ldots, f_n$ such that $f = f_0 + f_1 + \cdots + f_n$ and $g = f_0 + T_1(f_1) + \cdots + T_n(f_n)$.

Remarks

The usual notion of equidecomposition of sets can be obtained by requiring all functions to be characteristic functions, that is $\{0, 1\}$-valued.

In this case, we will say that *sets* A, B are equidecomposible.

When the transformations T_i come from a group action a, we will say that functions f, g are a-equidecomposible.

Flows

Let G be a graph on a vertex set V where we view the edge set E as symmetric irreflexive relation.

Let X_G be the set of functions $f : V \to \mathbb{R}$ and Φ_G be the set of functions $\phi : E \to \mathbb{R}$ such that for all $(x, y) \in E$, $\phi(x, y) = -\phi(y, x)$.

We define $\partial\phi = -\sum_{y \in N_G(x)} \phi(x, y)$ where $N_G(x)$ is the neighborhood of x in G.

We say that ϕ is an f-flow if $\partial\phi + f = 0$

Let $T_1, \ldots, T_n$ be transformations. Put $(x, y) \in E$ if and only if there is $i \leq n$ such that $T_i(x) = y$ or $T_i(y) = x$.

Proposition

For all $f, g \in \mathbb{R}^X$, there is an $f - g$-flow in (X, E) if and only if f and g are equidecomposible as functions using the transformations T_i for $i \leq m$ and the identity.

The following equations explain the proof: $f_i(x) = \phi(x, T_i(x))$ and $f_0 = f - \sum_{i=1}^{m} f_i$

Measures and paradoxes

We have the following theorem that combines work of Tarski and the definitions from the previous slides. For an action a, we write $G(a, S)$ for the Schreier graph of the action with generators from S.

Proposition

Suppose a is a Borel action of a countable group Γ on a standard Borel space X. Then the following are equivalent:

1. *There is no finitely additive invariant Borel measure μ on X taking values in $[0, \infty]$ such that $\mu(A) = 1$.*
2. *For some m, m copies of A are a-equidecomposable with $m + 1$ copies of A using Borel sets.*
3. *1_A is a-equidecomposible with 0 (the constantly 0 function) using bounded Borel integer-valued functions.*
4. *1_A is a-equidecomposible with 0 using bounded Borel functions.*

A sketch of the proof

1. (1) implies (2) is a theorem of Tarski. Assuming that (2) fails, a measure as in (1) can be constructed by transfinite induction.

2. (2) implies (3). (2) implies that $(m+1)1_A$ and $m1_A$ are equidecomposible where the witnessing functions are characteristic functions. By the equivalence between equidecompositions and flows, there is an integer valued flow of $(m+1)1_A - m1_A = 1_A$ in $G(a, S)$ for a suitably chosen S.

Proof sketch continued

1. (3) implies (4) is clear.

2. Assume (4) and there is a measure μ as in (1). By (4), there are $f_0, \dots, f_n$ and group elements $\gamma_1, \dots, \gamma_n$ such that $1_A = f_0 + \cdots + f_n$ and $0 = f_0 + \gamma_1(f_1) + \cdots + \gamma_n(f_n)$. We get a contradiction by integrating:

$$1 = \mu(A) = \int 1_A d\mu = \sum_{i=0}^{n} \int f_i d\mu =$$

$$\int f_0 d\mu + \sum_{i=1}^{n} \int \gamma(f_i) d\mu = \int 0 d\mu = 0$$

Approaches to Borel Ruziewicz

Proposition

Each of the following statements implies a positive answer to the Borel Ruziewicz problem.

1. *For* $n \geq 2$*, there is a function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{\epsilon \to 0^+} f(\epsilon) = 0$ *such that for all open sets* $A \subseteq \mathbb{S}_n$*, there are a finite set* $S \subseteq \mathrm{SO}_{n+1}$ *and a flow* ϕ *on* $G(a, S)$ *such that* $\partial\phi + 1_A$ *is bounded by* $f(\epsilon)$*.*
2. *For* $n \geq 2$*, for all open sets* $A \subseteq \mathbb{S}^n$*,* A *is equidecomposible by isometries to a ball.*

Circle squaring via flows

The setting:

- Work with "nice" sets $A, B \subseteq \mathbb{T}^k$ with the same Lebesgue measure.
- The goal is to find an equidecomposition of A, B via translations.

As an intermediate step, we will try to find an equidecomposition of 1_A and 1_B or equivalently a bounded flow of $1_A - 1_B$ in a graph generated by translations.

Completing the proof of circle squaring

There are two steps:

1. Convert the bounded $1_A - 1_B$-flow to bounded integer valued flow.

2. Convert the bounded integer valued flow to bijection from A to B using finitely many translations from the action.

Both of these steps can be made constructive. A recent theorem of Máthé-Noel-Pikhurko shows that the pieces of the decomposition can have "small boundary" assuming that A and B do.

Averaging

Let $u_1, \ldots, u_d \in \mathbb{T}^k$ and let a be the action of $\mathbb{Z}^d$ be the action given by translating by integer linear combinations of $u_1, \ldots, u_d$.

Let ρ be a finitely supported measure on $\mathbb{Z}^d$ and $f : \mathbb{T}^k \to \mathbb{R}$ be a bounded function.

Define $Av_\rho(f) : \mathbb{T}^k \to \mathbb{R}$ by $Av_\rho(f)(x) = \sum_{\gamma \in \mathbb{Z}^d} \rho(\gamma) f(\gamma \cdot x)$.

Examples

1. $\rho(\pm e_i) = 1/2d$ corresponds to a single step of the simple random walk. In this case, Av_ρ is a kind of graph Laplacian.

2. For $N \in \mathbb{N}$, ρ is the uniform probability measure over the set $\{\gamma \in \mathbb{Z}^d \mid 0 \leq \gamma < N \text{ pointwise } \}$. These measures appear in Laczkovich's solution to circle squaring.

Obtaining flows by averaging

Let $f : \mathbb{T}^k \to \mathbb{R}$ with $\int f d\lambda = 0$.

Let $\Delta = Av_\rho$ where ρ is the measure from the simple random walk. Consider the sequence

$$f, \Delta(f), \Delta^2(f), \ldots$$

which we hope converges to 0.

When we apply Δ to a function h, this is "implemented" by a flow ϕ_h in the sense that $\partial\phi_h + h = \Delta(h)$.

Obtaining flows by averaging

In particular for the sequence above, if we let ϕ_n be the flow such that $\partial\phi_n\Delta^n(f) = \Delta^{n+1}(f)$, then we have

$$\partial\Big(\sum_{n=0}^{\infty}\phi_n\Big) + f = \lim_{N\to\infty}\partial\Big(\sum_{n=0}^{N}\phi_n\Big) + f = \lim_{N\to\infty}\Delta^{N+1}(f) = 0$$

assuming that $\sum_{n=0}^{\infty}\phi_n$ is absolutely convergent.

A straightfoward calculation using the definition of ϕ_n, shows that the absolute convergence of $\sum_{n=0}^{\infty}\phi_n$ reduces to the absolute convergence of $\sum_{n=0}^{\infty}\Delta^n(f)$.

Analysis of $\Delta^n(f)$

Recall that we let ρ be the measure on $\mathbb{Z}^d$ corresponding to the simple random walk.

A straightforward calculation shows that $\Delta^n(f) = Av_{\rho^{*n}}(f)$ where ρ^{*n} is the convolution of ρ with itself n-times.

So our goal is to have a summable numerical bound on $|Av_{\rho^{*n}}(f)|$ in terms of n.

Note that for each x, $Av_{\rho^{*n}}(f)(x)$ is a finite average of values of f, so we will use techniques from numerical integration and discrepancy theory to bound this.

Averages using uniform measures

Let σ_n be the uniform probability measure over the set $\{\gamma \in \mathbb{Z}^d \mid 0 \leq \gamma < 2^n \text{ pointwise }\}$.

Consider the sequence

$$f, Av_{\sigma_1}(f), Av_{\sigma_2}(f), \ldots$$

This sequence also produces a flow which relies on the absolute convergence of $\sum_{n=0}^{\infty} 2^n Av_{\sigma_n}(f)$ and which has a more complicated formula.