Does $\mathcal{P}(\omega)/\mathrm{Fin}$ know its right hand from its left?

Part 2

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January 27, 2025

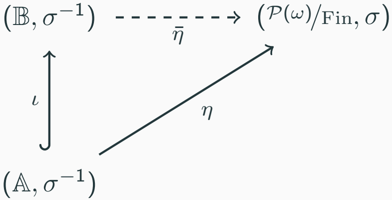
University of North Carolina at Charlotte

Statement of the lemma

Recall from the last talk the statement of the key lemma:

Lemma (the Lifting Lemma)

Let $(\mathbb{A}, \sigma^{-1})$ and $(\mathbb{B}, \sigma^{-1})$ be countable substructures of $({}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma^{-1})$ with $\mathbb{A} \subseteq \mathbb{B}$, and suppose η is an elementary embedding from $(\mathbb{A}, \sigma^{-1})$ into $({}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma)$. Then η extends to an embedding $\bar{\eta}$ of $(\mathbb{B}, \sigma^{-1})$ into $({}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}, \sigma)$, with $\bar{\eta} \circ \iota = \eta$.



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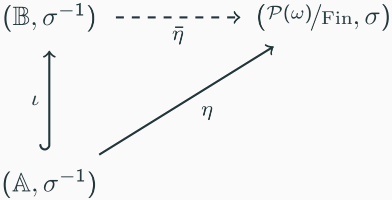


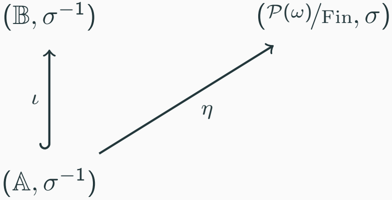
The goal of this talk is to discuss some of the ideas that go into the proof of this lemma.

Restatement of the lemma

An *instance of the lifting problem* is a 4-tuple

$$\big((\mathbb{A},\sigma^{-1}),(\mathbb{B},\sigma^{-1}),\iota,\eta\big)$$

where $\mathbb{A},\mathbb{B}$ are countable subalgebras of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$ closed wrt σ,σ^{-1}, and $\mathbb{A}\subseteq\mathbb{B}$, and η is an embedding $(\mathbb{A},\sigma^{-1})\to({}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}},\sigma)$.



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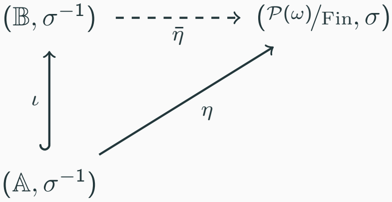

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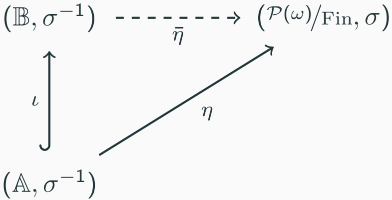

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Lifting Lemma: An instance $\big((\mathbb{A},\sigma^{-1}),(\mathbb{B},\sigma^{-1}),\iota,\eta\big)$ of the lifting problem has a solution if η is an elementary embedding.

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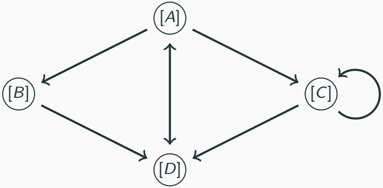

$A = \{n \in \omega \colon n \text{ ends in a } 0, 3, \text{ or } 5\}$

$B = \{n \in \omega \colon n \text{ ends in a } 1\}$

$C = \{n \in \omega \colon n \text{ ends in a } 6, 7, \text{ or } 8\}$

$D = \{n \in \omega \colon n \text{ ends in a } 2, 4, \text{ or } 9\}$

Specifically, by strongly connected digraphs

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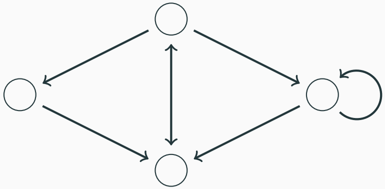

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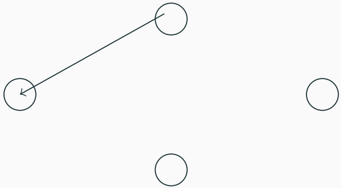

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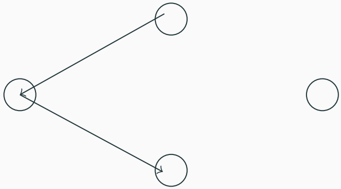

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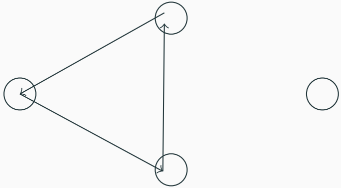

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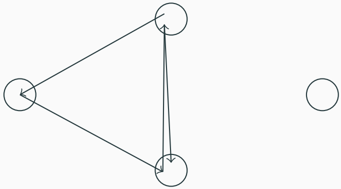

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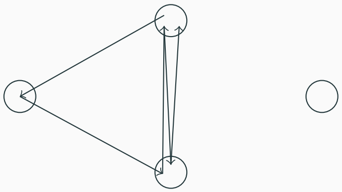

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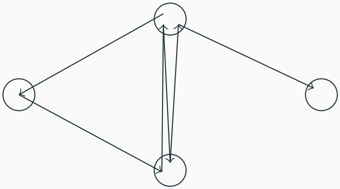

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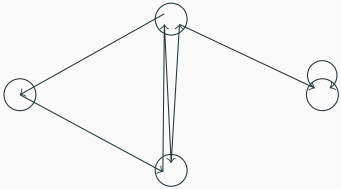

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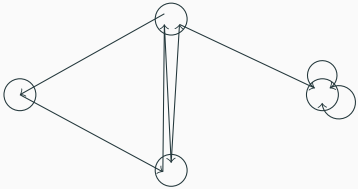

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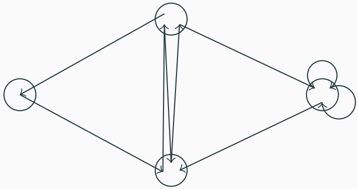

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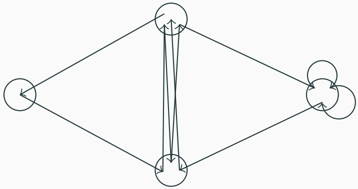

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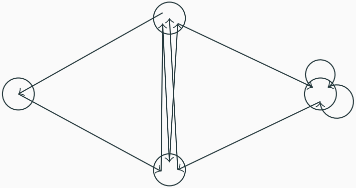
For each $i < k$, put $A_i = \{n \colon n \equiv i \pmod{k}\} \in \mathcal{A}$. □



Finer partitions = richer digraphs

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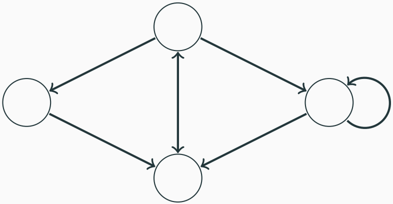

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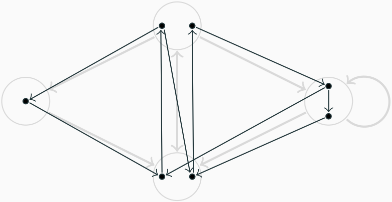

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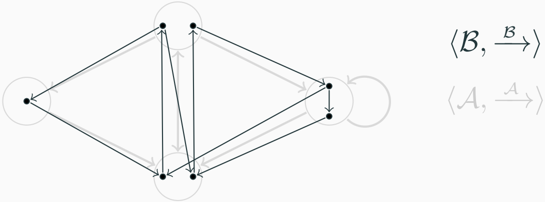

Lemma

Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite partitions of $\mathcal{P}(\omega)/\mathrm{Fin}$. If $\mathcal{B}$ is a refinement of $\mathcal{A}$, then the natural mapping $\mathcal{B} \to \mathcal{A}$ is an epimorphism from $\langle \mathcal{B}, \xrightarrow{\mathcal{B}} \rangle$ to $\langle \mathcal{A}, \xrightarrow{\mathcal{A}} \rangle$.

Not all epimorphisms correspond to refinements

Given a finite partition $\mathcal{A}$ of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$ and the corresponding digraph $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$, not all epimorphisms correspond to refinements of $\mathcal{A}$.

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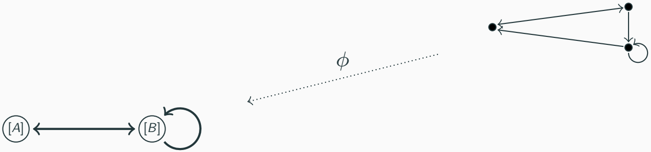

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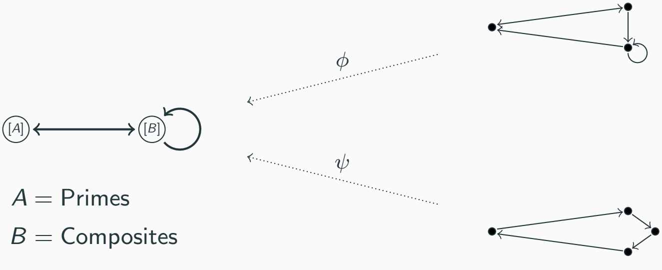

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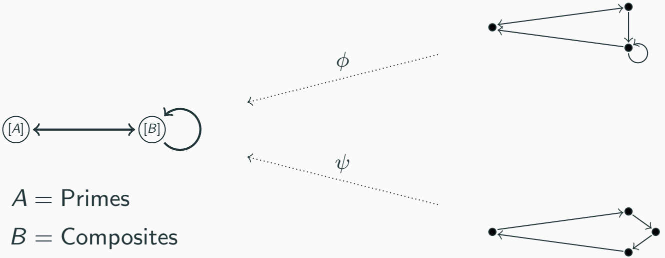

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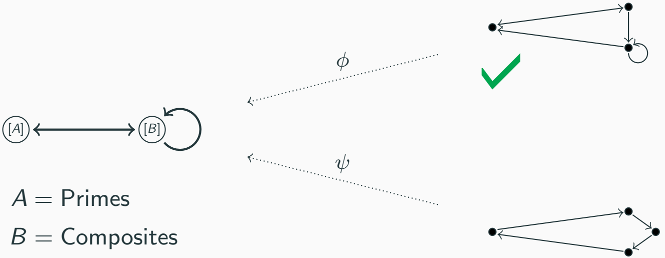

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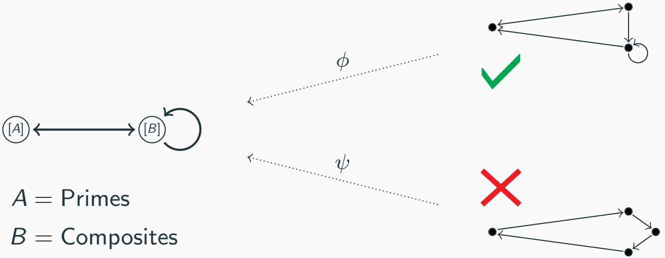

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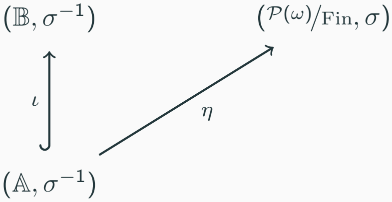
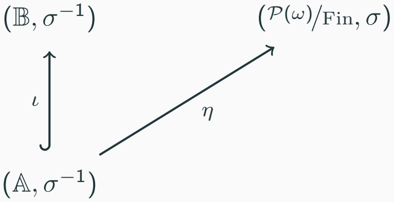
Back to the lemma



Given an instance of the lifting problem, the countable structures $(\mathbb{A}, \sigma^{-1})$ and $(\mathbb{B}, \sigma^{-1})$ can be approximated by sequences of finite digraphs as on the previous slide.

Back to the lemma



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The embedding η translates each finite partition $\mathcal{A}$ of $\mathbb{A}$ into a partition $\tilde{\mathcal{A}}$ of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$, and the resulting digraphs are isomorphic:

$$\langle \mathcal{A}, \xrightarrow{\sigma^{-1}} \rangle \cong \langle \tilde{\mathcal{A}}, \xrightarrow{\sigma} \rangle \qquad \text{where} \qquad \tilde{\mathcal{A}} = \{\eta(a) : a \in \mathcal{A}\}.$$

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$$
\begin{array}{ccc}
(\mathbb{B}, \sigma^{-1}) & \xrightarrow[\bar{\eta}]{} & \left(\mathcal{P}(\omega)/\mathrm{Fin}, \sigma\right) \\
\iota \big\uparrow & \nearrow_{\eta} & \\
(\mathbb{A}, \sigma^{-1}) & &
\end{array}
$$

$$
\begin{array}{ccc}
\langle \mathcal{B}, \xrightarrow{\sigma^{-1}} \rangle & & \\
\Big\downarrow \text{natural epimorphism} & & \\
\langle \mathcal{A}, \xrightarrow{\sigma^{-1}} \rangle & \underset{\text{induced by } \eta}{\cong} & \langle \tilde{\mathcal{A}}, \xrightarrow{\sigma} \rangle
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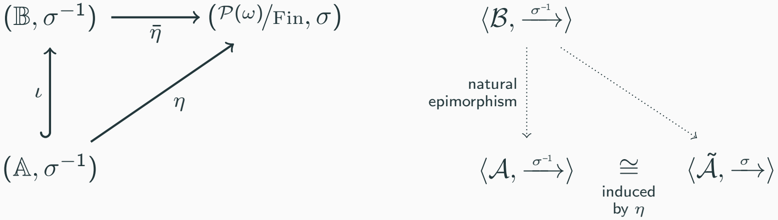

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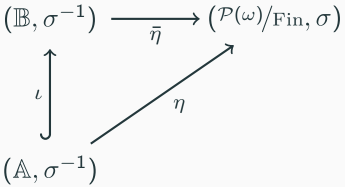

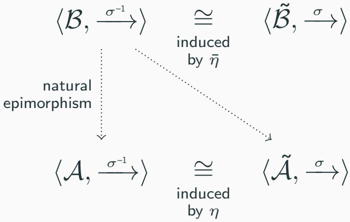

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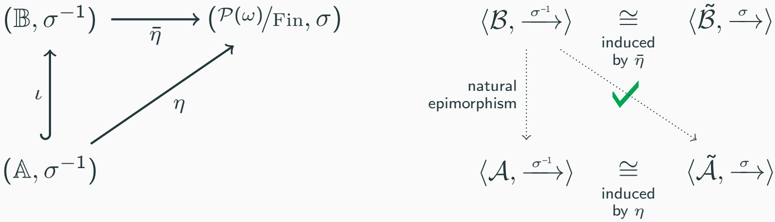

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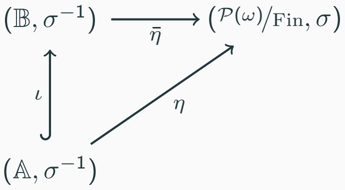

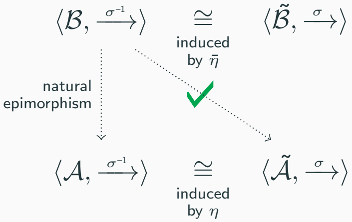

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Lemma

The converse is also true: If all the epimorphisms arising in this way via ι and η are realizable, then $\bar{\eta}$ exists.

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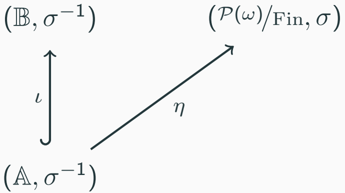

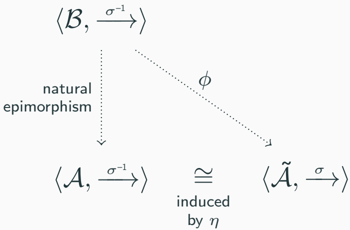

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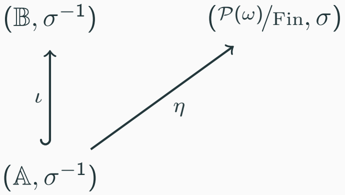

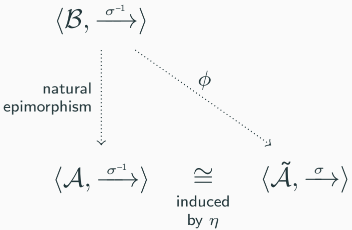

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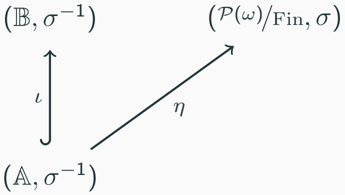

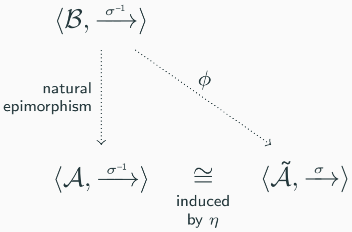

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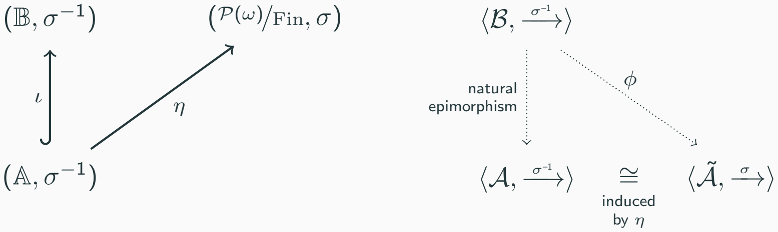

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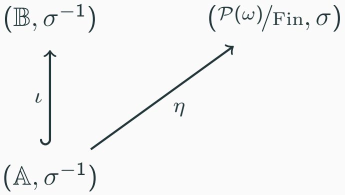

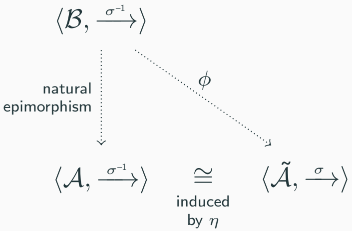

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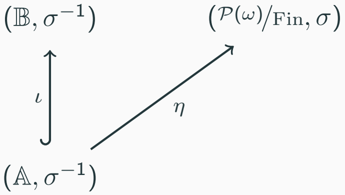

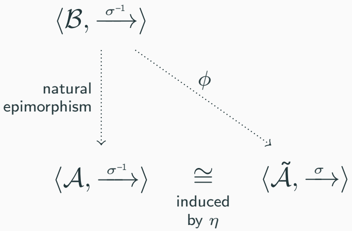

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Incompatible epimorphisms

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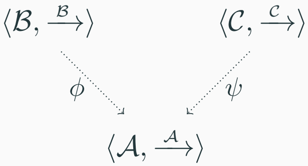

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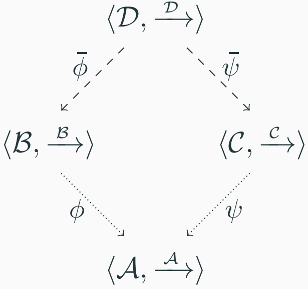

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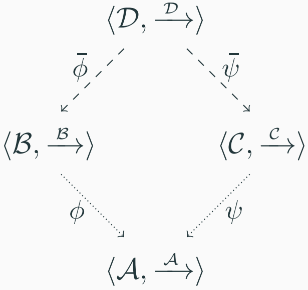
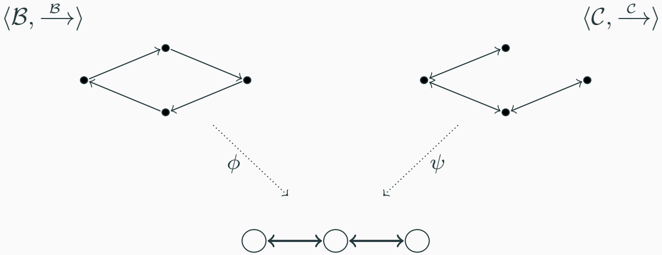


Otherwise ϕ and ψ are *incompatible*.

An example of incompatible epimorphisms



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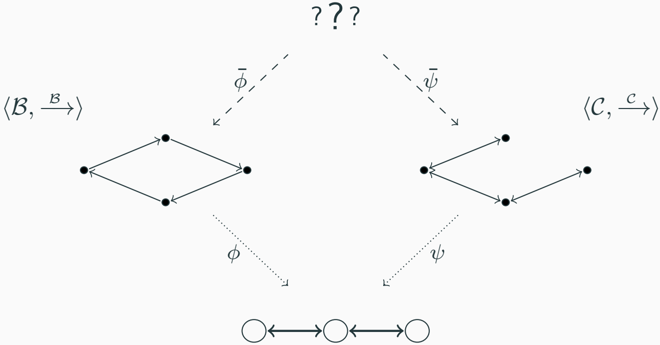

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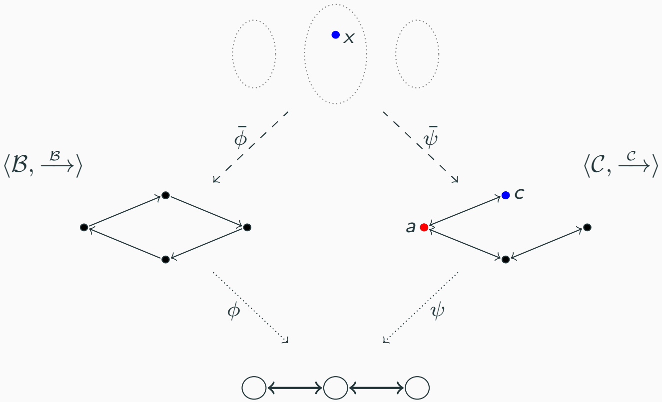

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Let $x \in \bar{\psi}^{-1}(c)$.

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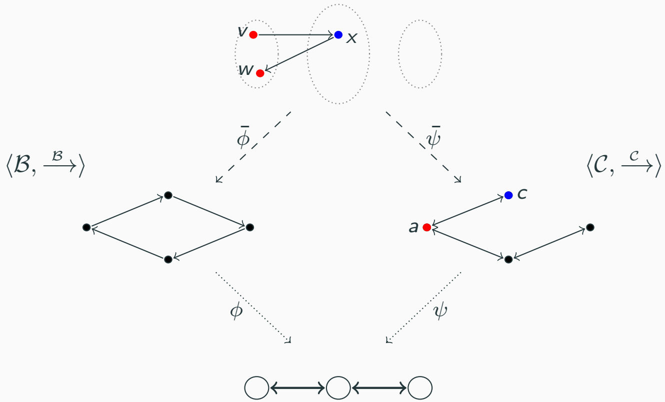

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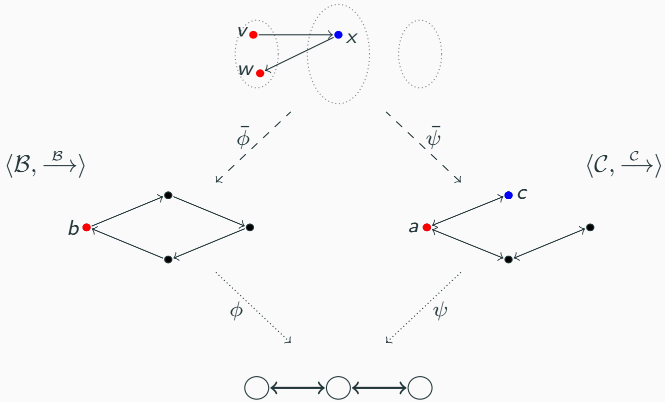

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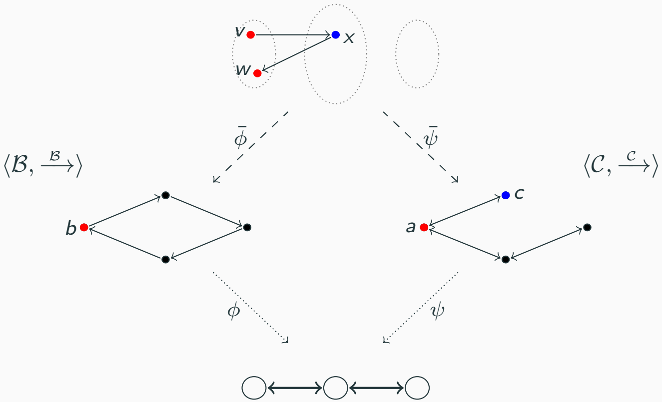

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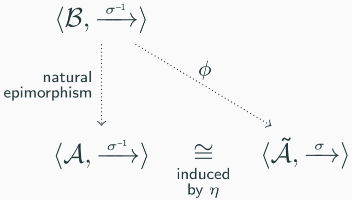
Because $\bar{\phi}(v) = \bar{\phi}(w) = b$, we should have $b \to \bar{\phi}(x) \to b$.
But $\langle \mathcal{B}, \xrightarrow{\mathcal{B}} \rangle$ contains no such vertex.

The relevant epimorphisms are always compatible.



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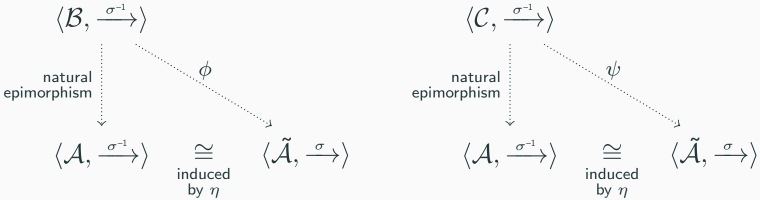

Lemma

Given a partition $\mathcal{A}$ of $\mathbb{A}$ and its image $\tilde{\mathcal{A}}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$, any two epimorphisms arising naturally in the Lifting Lemma (any two finitary instances of the lifting problem) are compatible.

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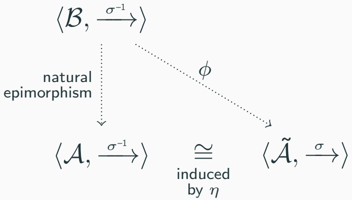

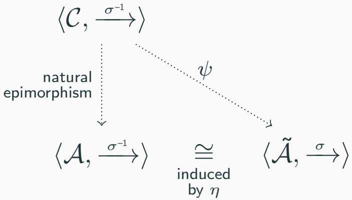

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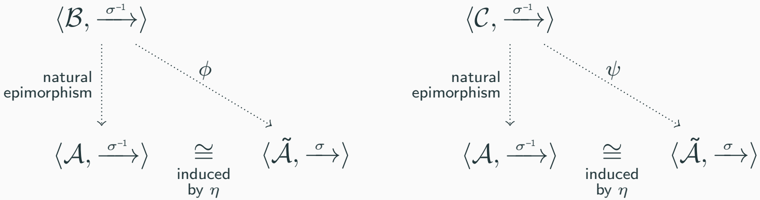

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In other words, the epimorphisms that we actually encounter in the lifting problem are always compatible with one another.

A dichotomy for finite fragments of $\left({}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}}, \sigma\right)$

Theorem (Dichotomy Theorem)

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Let $\mathcal{A}$ be a finite partition of ${}^{\mathcal{P}(\omega)}/_{\mathrm{Fin}}$, and let ϕ be an epimorphism from a strongly connected digraph $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$ to $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$.

$$\begin{array}{c} \langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle \\ \Big\downarrow \phi \\ \langle \mathcal{A}, \xrightarrow{\sigma} \rangle \end{array}$$

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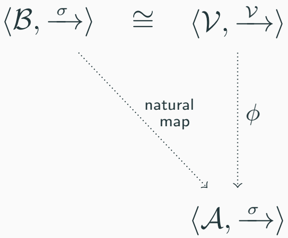

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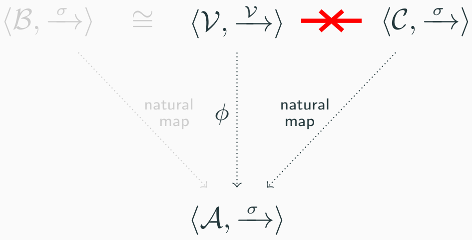

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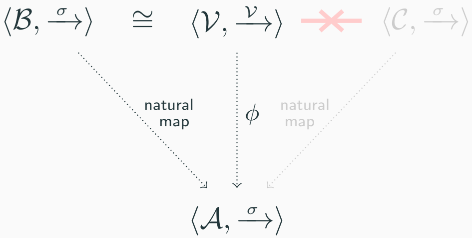
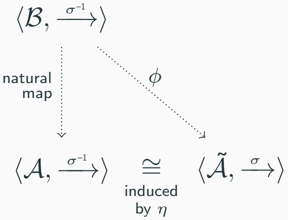


The role of elementarity

Consider some finitary instance of the lifting problem.



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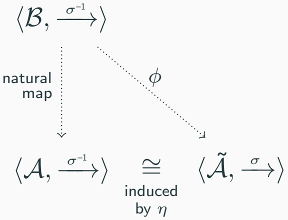

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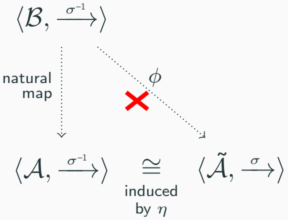

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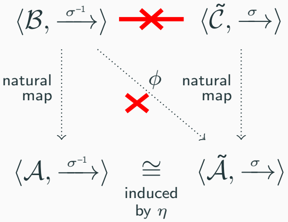

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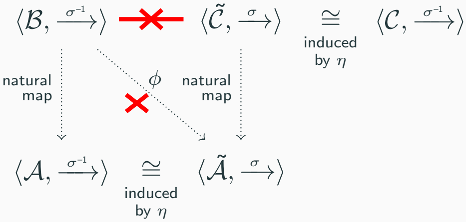

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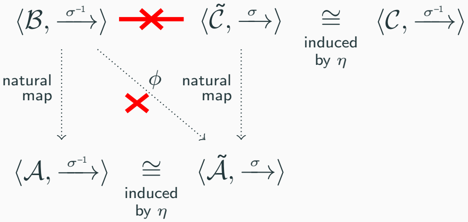

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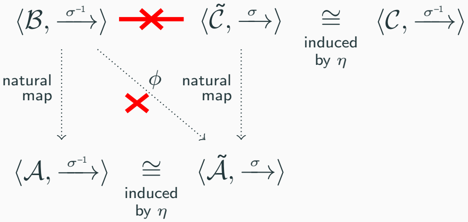

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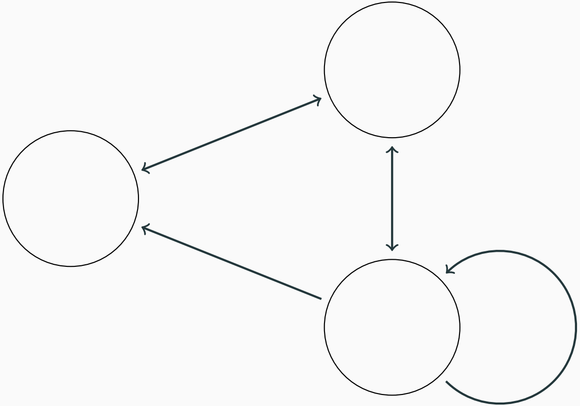
But how do you prove the Dichotomy Theorem?

To get some idea of the proof of the Dichotomy Theorem, fix a partition $\mathcal{A}$ of ${}^{\mathcal{P}(\omega)}\!/_{\mathrm{Fin}}$



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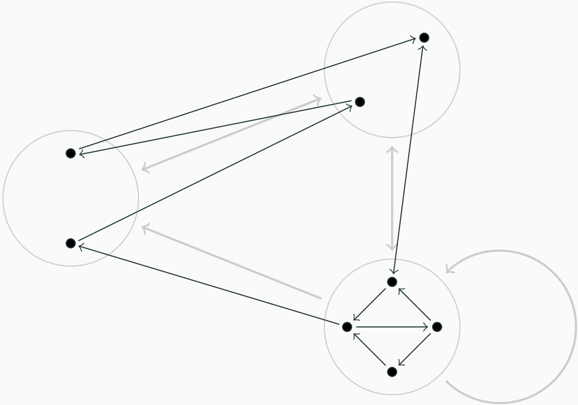

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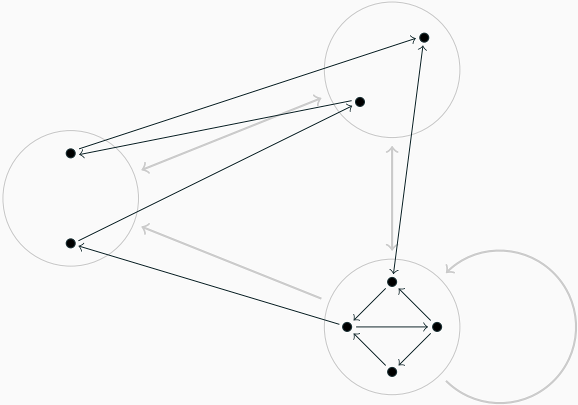
Suppose alternative 1 fails: i.e., there is no refinement $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{P}(\omega)/_{\mathrm{Fin}}$ such that the natural map $\mathcal{B} \to \mathcal{A}$ mimics ϕ.



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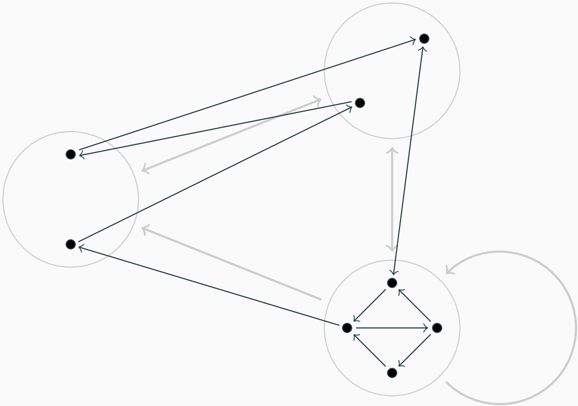
The partition $\mathcal{A}$ is not recoverable from the isomorphism class of its digraph $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$.



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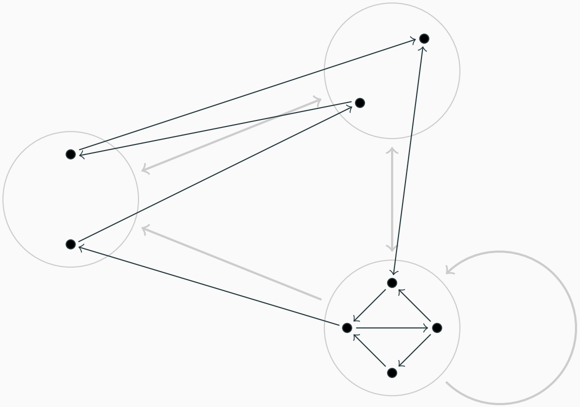
The partition $\mathcal{A}$ is not recoverable from the isomorphism class of its digraph $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$. But it can be recovered from $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$ plus a specific infinite walk through $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$.



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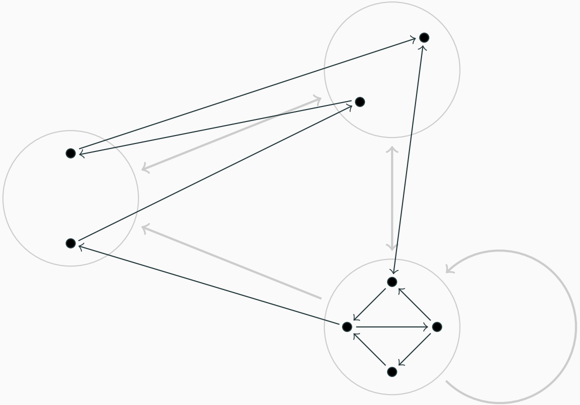
The failure of alternative 1 means that there is no infinite walk through $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$ that follows our walk through $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$, or even that follows it with finitely many errors.



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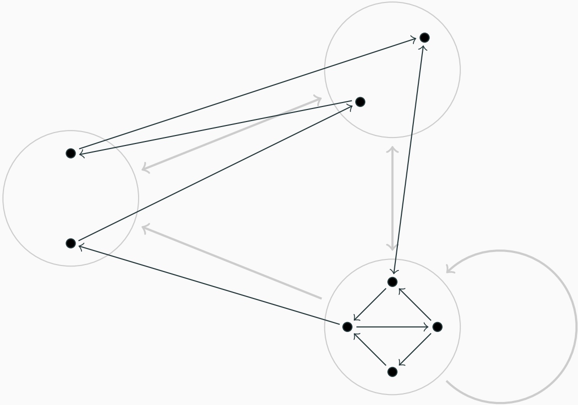
In other words, if we imagine someone walking through $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$, trying to follow our specific walk in $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$, then they get "lost" after finitely many steps, regardless of when they started following.



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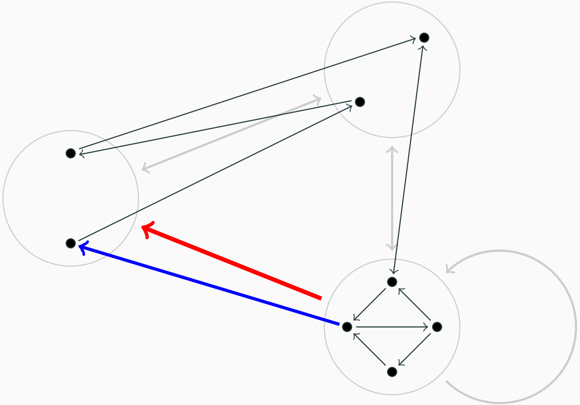

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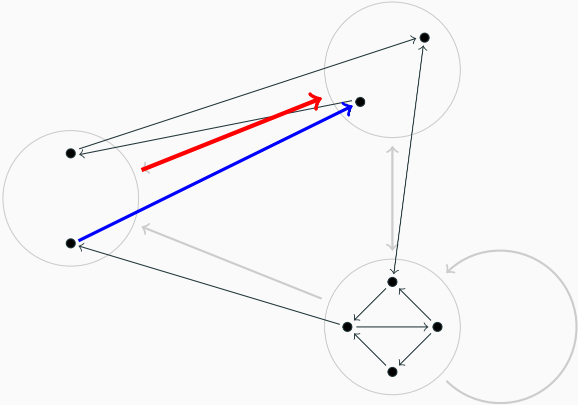

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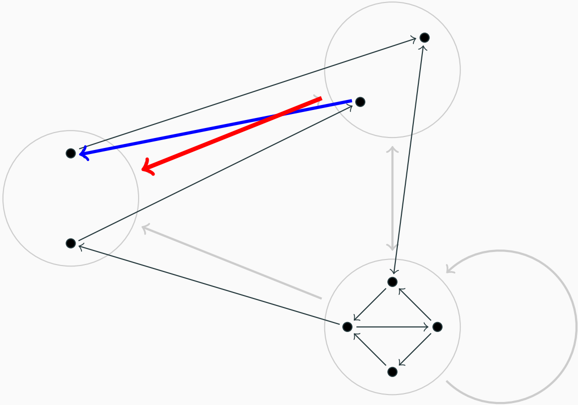

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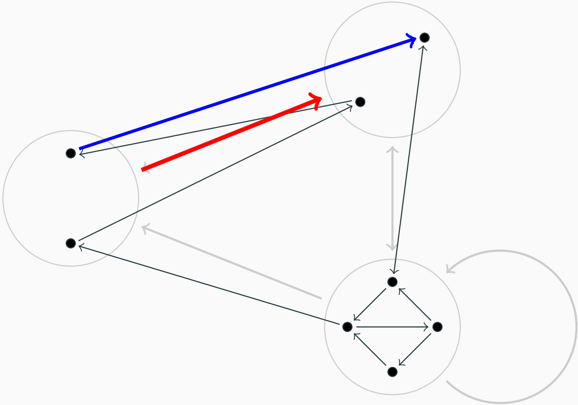

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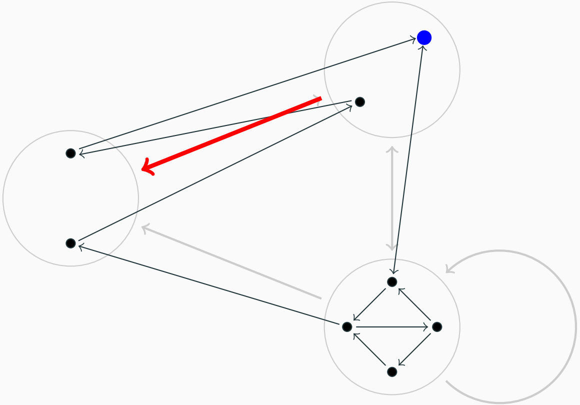

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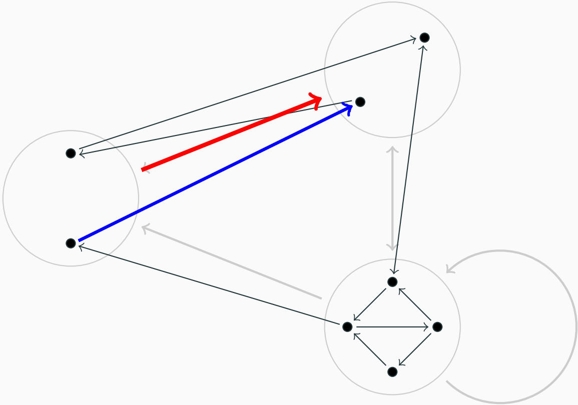

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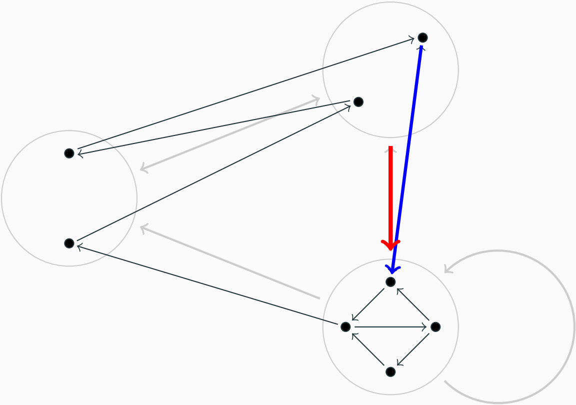

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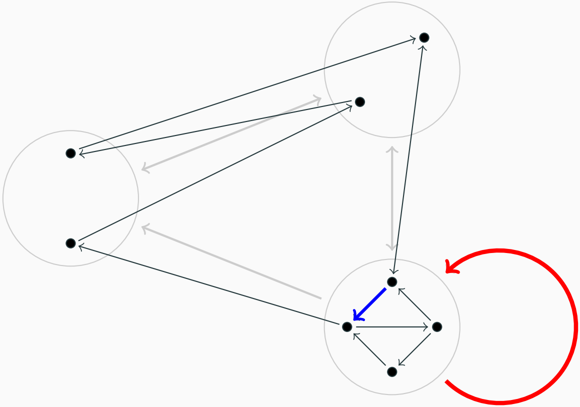

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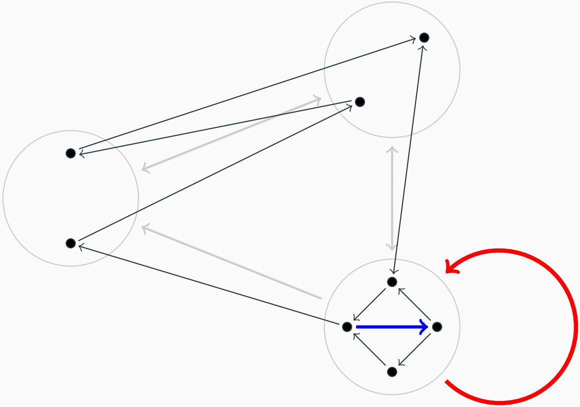

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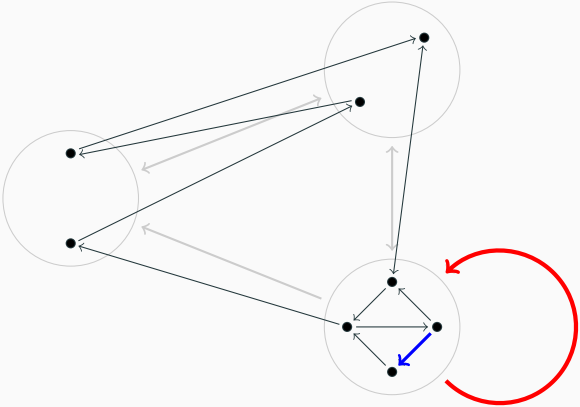

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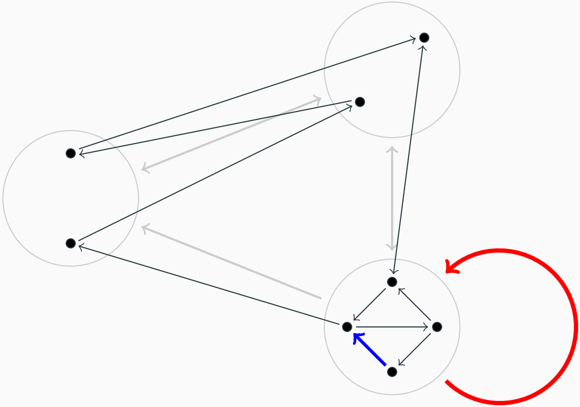

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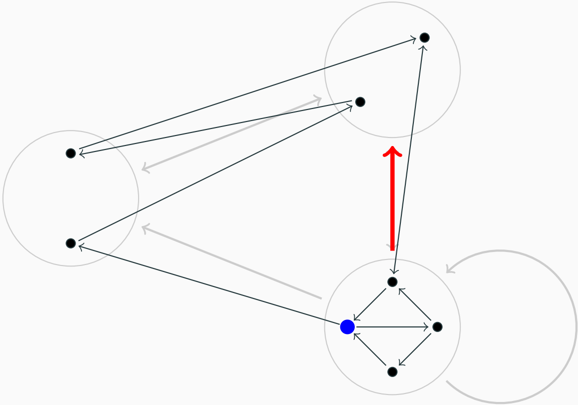

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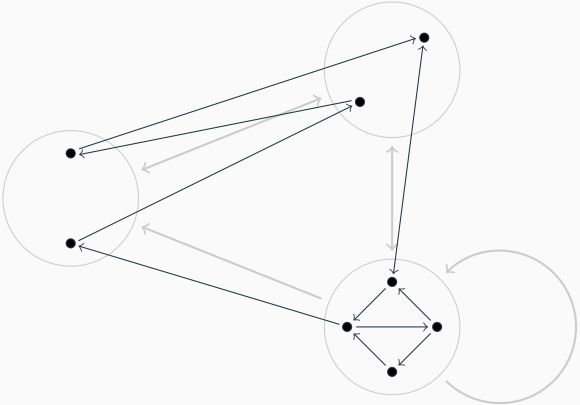
Using this, we must find a partition $\mathcal{C}$ of $\mathcal{A}$ such that the natural map from $\langle \mathcal{C}, \xrightarrow{\sigma} \rangle$ to $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$ is incompatible with ϕ.



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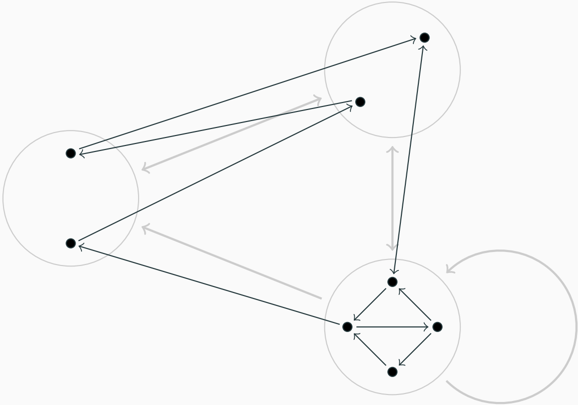
The rough idea is that the finite digraph $\langle \mathcal{C}, \xrightarrow{\sigma} \rangle$ must encode all the different ways in which a walker in $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$ can get lost.



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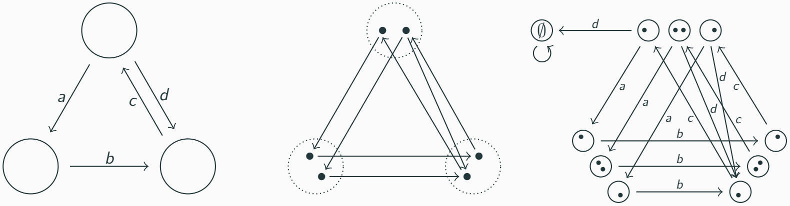
An important piece of $\mathcal{C}$ is the *state space* digraph arising from ϕ.



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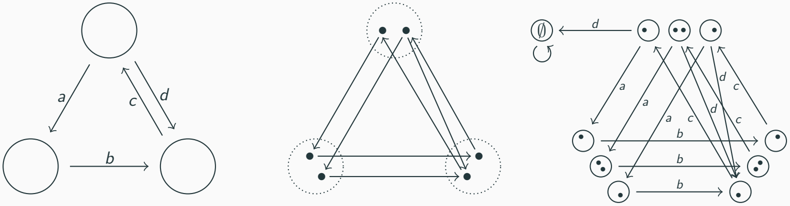

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Roughly, this digraph keeps track not of the individual vertices in a walk through $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$, but the set of possible vertices where a "follower" might be at a given time.

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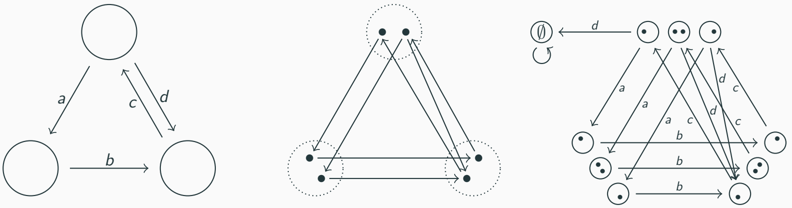

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Roughly, this digraph keeps track not of the individual vertices in a walk through $\langle \mathcal{V}, \xrightarrow{\mathcal{V}} \rangle$, but the set of possible vertices where a "follower" might be at a given time. The digraph $\langle \mathcal{C}, \xrightarrow{\sigma} \rangle$ combines this state space with an isomorphic copy of $\langle \mathcal{A}, \xrightarrow{\sigma} \rangle$.

Thank you for listening!

Any questions?