Uncountable ultrahomogeneous structures II

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The setup

- We fix a countable *(ultra)homogeneous* structure U, meaning that every isomorphism $f: A \to B$ between its finitely generated substructures can be extended to an automorphism.
- Let $\mathcal{F}$ be the *age* of U, i.e. the class of all finitely generated structures embeddable into U.
- Then $\mathcal{F}$ is a *Fraïssé class*, i.e. it is hereditary, has countably many isomorphism-types, has the joint embedding property, and the *amalgamation property*.
- For every Fraïssé class $\mathcal{F}$ there is a unique countable homogeneous structure U with age $\mathcal{F}$.

Main question

Does there exist an uncountable homogeneous structure V with age $\mathcal{F}$?

Classical positive cases

- Let $\sigma\mathcal{F}$ denote the class of all countable unions of chains from $\mathcal{F}$, equivalently of all countable structures with age $\subseteq \mathcal{F}$.
- (CH) If $\sigma\mathcal{F}$ has the amalgamation property, then there is a unique $\sigma\mathcal{F}$-homogeneous and universal structure V of size ω_1 (*Fraïssé–Jónsson limit*).
- But that is much stronger condition. We are interested in existence of V that is homogeneous only for finitely generated structures, but under much more general assumptions.

- If $\mathcal{F}$ is in finite relational language, then the ultrapower U^ω/p for a non-principal ultrafilter p is homogeneous with age $\mathcal{F}$.
- But in general, ultrapowers may enlarge the age.

Degenerate negative cases

- A necessary condition is that there is a *non-trivial* self-embedding of U, i.e. $\mathrm{Emb}(U) \supsetneq \mathrm{Aut}(U)$.
- This excludes finite homogeneous structures, rigid homogeneous structures, and more:

Example

- Let $\mathcal{F}$ be the class of torsion-free cyclic groups: $\{0, \mathbb{Z}\}$.
- Then $\sigma\mathcal{F}$ consists of all torsion-free locally cyclic groups, and $U = \mathbb{Q}$.
- $\mathbb{Q}$ does not have any non-trivial locally cyclic extension, and so $\mathrm{Emb}(U) = \mathrm{Aut}(U)$.

Extension property

- Recall that a structure X has the *extension property* or is *injective* if for every $A \subseteq B$ in $\mathcal{F}$ and every embedding $f\colon A \to X$ there is an extension $\tilde{f}\colon B \to X$.
- Every universal homogeneous structure is injective, and in the countable case the converse holds as well.
- A linear order is injective if and only if it is dense, but for example $\mathbb{R} \oplus \mathbb{R}$ is not homogeneous.
- There is an uncountable injective structure V with age $\mathcal{F}$ if (and only if) $\mathrm{Emb}(U) \supsetneq \mathrm{Aut}(U)$: take the colimit of ω_1-many iterations of a fixed non-trivial embedding $e\colon U \to U$.

$$U \overset{e}{\hookrightarrow} U \overset{e}{\hookrightarrow} \cdots \quad U_\omega = U \overset{e}{\hookrightarrow} U \overset{e}{\hookrightarrow} \cdots \quad U_{\omega_1} = V$$

- But how to get homogeneity?

Extensible embeddings

Definition

An embedding $e\colon X \to Y$ is *extensible* if for every $g \in \mathrm{Aut}(X)$ there is $h \in \mathrm{Aut}(Y)$ with $h \circ e = e \circ g$.

- If there is a non-trivial extensible embedding $e\colon U \to U$, then the ω_1-iteration works.
- Where to get it? Sometimes it is easy: e.g. the linear order embeddings $\mathbb{Q}_0 \to \mathbb{Q}_0 \oplus \mathbb{Q}_1$ or $\mathbb{Q}_0 \to \mathbb{Q}_0 \times_{\mathrm{lex}} \mathbb{Q}_1$.
- More generally, if there is a *Katětov functor* $K\colon \mathcal{F} \to \sigma\mathcal{F}$ (see Kubiś–Mašulović '17), then $\mathrm{Aut}(U)$ is universal for $\{\mathrm{Aut}(X) : X \in \sigma\mathcal{F}\}$, and U has a non-trivial extensible self-embedding.

Definition

For $G \subseteq \mathrm{Aut}(U)$ an embedding $e\colon X \to Y$ is *G-extensible* if for every $g \in G$ there is $h \in \mathrm{Aut}(Y)$ with $h \circ e = e \circ g$.

What is really needed

Definition

An *extensible U-chain* is a sequence $\langle U_\alpha, G_\alpha, s_\alpha^\beta \rangle_{\alpha<\beta<\omega_1}$ such that

1. $\langle U_\alpha \rangle_{\alpha<\omega_1}$ is a strictly increasing continuous chain of isomorphic copies of U,
2. every G_α is a countable family of automorphisms witnessing the homogeneity of U_α,
3. for every $\alpha < \beta < \omega_1$ the map $s_\alpha^\beta \colon G_\alpha \to G_\beta$ witnesses that the inclusion $U_\alpha \subseteq U_\beta$ is G_α-extensible,
4. the system $\langle s_\alpha^\beta \rangle_{\alpha<\beta<\omega_1}$ is coherent, i.e. $s_\beta^\gamma \circ s_\alpha^\beta = s_\alpha^\gamma$.

$$U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots U_\omega \subsetneq U_{\omega+1} \subsetneq \cdots \qquad U_{\omega_1}$$

$$G_0 \xrightarrow[s_0^1]{} G_1 \xrightarrow[s_1^2]{} G_2 \xrightarrow[s_2^\omega]{} G_\omega \xrightarrow[s_\omega^{\omega+1}]{} G_{\omega+1} \longrightarrow$$

Main results

$$U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots U_\omega \subsetneq U_{\omega+1} \subsetneq \cdots \qquad U_{\omega_1}$$

$$G_0 \xrightarrow[s_0^1]{} G_1 \xrightarrow[s_1^2]{} G_2 \xrightarrow[s_2^\omega]{} G_\omega \xrightarrow[s_\omega^{\omega+1}]{} G_{\omega+1} \longrightarrow$$

Theorem

If $\langle U_\alpha, G_\alpha, s_\alpha^\beta \rangle_{\alpha<\beta<\omega_1}$ is an extensible U-chain, then $U_{\omega_1} = \bigcup_{\alpha<\omega_1} U_\alpha$ is homogeneous of cardinality ω_1.

Corollary

There is an uncountable homogeneous structure V with age $\mathcal{F}$ if and only if there is an extensible U-chain.

Theorem

If U has a non-trivial *G-extensible embedding for every countable* $G \subseteq \mathrm{Aut}(U)$, then there is an extensible U-chain.

Sufficient conditions

The following implications are easy or well-known:

$\mathcal{F}$ is in finite relational language

$\Rightarrow$ U is ω-categorical

$\Rightarrow$ U is ω-saturated

$\Rightarrow$ *$\sigma\mathcal{F}$ has the amalgamation property*

$\Rightarrow$ U is an amalgamation base, or equivalently the monoid $\mathrm{Emb}(U)$ has the amalgamation property.

Theorem

If $\mathrm{Emb}(U) \supsetneq \mathrm{Aut}(U)$ has the amalgamation property, then for every countable $G \subseteq Aut(U)$ there is a non-trivial embedding $e\colon U \to U$.

Proof sketch

We let e be the infinite composition along a Fraïssé sequence in a countable monoid $M \subseteq \mathrm{Emb}(U)$ containing G, containing a non-trivial embedding, and having the amalgamation property.

Main example

- Jan Grebík has considered the Fraïssé class $\mathcal{F}$ of all finite linearly ordered ω-edge-colored complete graphs without monochromatic triangles, i.e. structures $\langle X, c_X, <_X \rangle$ where X is finite, $c_X \colon [X]^2 \to \omega$ has no monochromatic triangles, and $<_X$ is a linear order.
- He proved that $\mathrm{Aut}(U)$ is universal, but there is no Katětov functor $\mathcal{F} \to \sigma\mathcal{F}$.
- It is easy to see that U is not an amalgamation base.
- We showed that U has a non-trivial G-extensible embedding for every countable $G \subseteq \mathrm{Aut}(U)$.
- So there is an uncountable homogeneous structure with age $\mathcal{F}$ even though $\sigma\mathcal{F}$ does not have the amalgamation property and $\mathcal{F}$ does not have a Katětov functor.

Questions

We have the following implications:

U has a non-trivial G-extensible embedding for every countable $G \subseteq \mathrm{Aut}(U)$

$\Rightarrow$ there is an extensible U-chain of length ω_1

$\Rightarrow$ there is an extensible U-chain of length 2, i.e. a non-trivial embedding $e\colon U \to U$ such that every isomorphism of finitely generated substructures can be extended to an automorphism simultaneously on both levels,

$\Rightarrow$ $\mathrm{Emb}(U) \supsetneq \mathrm{Aut}(U)$.

But are they reversible?

In particular, supposing $\mathrm{Emb}(U) \supsetneq \mathrm{Aut}(U)$, is there always an uncountable homogeneous structure with age $\mathcal{F}$?

Uncountable homogeneous structures [arXiv:2411.17889]