Aspects of iterated forcing

Jörg Brendle

Kobe University

January/February 2010

Suslin ccc forcing

A p.o. $\mathbb{P}$ is called a *Suslin ccc forcing notion* if it is ccc and

$$\mathbb{P} \subseteq \omega^\omega,$$
$$\leq_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega, \text{ and}$$
$$\perp_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega$$

are all analytic sets.

Suslin ccc forcing

A p.o. $\mathbb{P}$ is called a *Suslin ccc forcing notion* if it is ccc and

$$\mathbb{P} \subseteq \omega^\omega,$$
$$\leq_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega, \text{ and}$$
$$\perp_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega$$

are all analytic sets.

Assume $M \models ZFC$. If the parameters in the definition of $\mathbb{P}$, $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ are in M, we may interpret $\mathbb{P}$ in M. Denote this interpretation by $\mathbb{P}^M$.

Suslin ccc forcing

A p.o. $\mathbb{P}$ is called a *Suslin ccc forcing notion* if it is ccc and

$$\mathbb{P} \subseteq \omega^\omega,$$
$$\leq_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega, \text{ and}$$
$$\perp_{\mathbb{P}} \subseteq \omega^\omega \times \omega^\omega$$

are all analytic sets.

Assume $M \models ZFC$. If the parameters in the definition of $\mathbb{P}$, $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ are in M, we may interpret $\mathbb{P}$ in M. Denote this interpretation by $\mathbb{P}^M$.

Assume $M \subseteq N$. By $\mathbf{\Sigma}^1_1$ absoluteness, the statements $p \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ and $p \perp_{\mathbb{P}} q$ are absolute between M and N.

Examples for Suslin ccc forcing 1

Hechler forcing $\mathbb{D}$:

- Conditions: pairs (s, f) with $f \in \omega^\omega$ and $s \subseteq f$ finite
- Order: $(t, g) \leq (s, f)$ if $t \supseteq s$ and $g \geq f$ (everywhere)

Examples for Suslin ccc forcing 1

Hechler forcing $\mathbb{D}$:

- Conditions: pairs (s, f) with $f \in \omega^\omega$ and $s \subseteq f$ finite
- Order: $(t, g) \leq (s, f)$ if $t \supseteq s$ and $g \geq f$ (everywhere)

Properties:

- σ-centered (thus ccc)

Examples for Suslin ccc forcing 1

Hechler forcing $\mathbb{D}$:

- Conditions: pairs (s, f) with $f \in \omega^\omega$ and $s \subseteq f$ finite
- Order: $(t, g) \leq (s, f)$ if $t \supseteq s$ and $g \geq f$ (everywhere)

Properties:

- σ-centered (thus ccc)
- adds a generic *Hechler real*

$$d = \bigcup \{s : \text{ there is } f \in \omega^\omega \text{ such that } (s, f) \in G\}$$

Examples for Suslin ccc forcing 1

Hechler forcing $\mathbb{D}$:

- Conditions: pairs (s, f) with $f \in \omega^\omega$ and $s \subseteq f$ finite
- Order: $(t, g) \leq (s, f)$ if $t \supseteq s$ and $g \geq f$ (everywhere)

Properties:

- σ-centered (thus ccc)
- adds a generic *Hechler real*

$$d = \bigcup \{s : \text{ there is } f \in \omega^\omega \text{ such that } (s, f) \in G\}$$

- d is a *dominating real*,
  i.e. $f \leq^* d$ for every $f \in \omega^\omega$ from the ground model.

Examples for Suslin ccc forcing 2

Check $\mathbb{D}$ is Suslin ccc:
identify $\mathbb{D}$ with $\omega \times \omega^\omega \cong \omega^\omega$ via $(s, f) \mapsto (|s|, f)$.
Then:

Examples for Suslin ccc forcing 2

Check $\mathbb{D}$ is Suslin ccc:
identify $\mathbb{D}$ with $\omega \times \omega^\omega \cong \omega^\omega$ via $(s, f) \mapsto (|s|, f)$.
Then:

- the order is a closed relation

Examples for Suslin ccc forcing 2

Check $\mathbb{D}$ is Suslin ccc:
identify $\mathbb{D}$ with $\omega \times \omega^\omega \cong \omega^\omega$ via $(s, f) \mapsto (|s|, f)$.
Then:

- the order is a closed relation
- (s, f) and (t, g) are incompatible iff

Examples for Suslin ccc forcing 2

Check $\mathbb{D}$ is Suslin ccc:
identify $\mathbb{D}$ with $\omega \times \omega^\omega \cong \omega^\omega$ via $(s, f) \mapsto (|s|, f)$.
Then:

- the order is a closed relation
- (s, f) and (t, g) are incompatible iff
  - either s and t are incomparable (a clopen relation)

Examples for Suslin ccc forcing 2

Check $\mathbb{D}$ is Suslin ccc:
identify $\mathbb{D}$ with $\omega \times \omega^\omega \cong \omega^\omega$ via $(s, f) \mapsto (|s|, f)$.
Then:

- the order is a closed relation
- (s, f) and (t, g) are incompatible iff
  - either s and t are incomparable (a clopen relation)
  - or one extends the other, say $s \subseteq t$ for simplicity, and $t(n) < f(n)$ for some n (again a clopen relation).

Examples for Suslin ccc forcing 3

Amoeba forcing $\mathbb{A}$:

- Conditions: open sets $U \subseteq 2^\omega$ of measure less than $\frac{1}{2}$
- Order: $V \leq U$ iff $V \supseteq U$

Examples for Suslin ccc forcing 3

Amoeba forcing $\mathbb{A}$:

- Conditions: open sets $U \subseteq 2^\omega$ of measure less than $\frac{1}{2}$
- Order: $V \leq U$ iff $V \supseteq U$

Properties:

- σ-linked (thus ccc)

Examples for Suslin ccc forcing 3

Amoeba forcing $\mathbb{A}$:

- Conditions: open sets $U \subseteq 2^\omega$ of measure less than $\frac{1}{2}$
- Order: $V \leq U$ iff $V \supseteq U$

Properties:

- σ-linked (thus ccc)
- adds an open subset $a = \bigcup G$ of 2^ω of measure $\frac{1}{2}$ (an *amoeba real*)

Examples for Suslin ccc forcing 3

Amoeba forcing $\mathbb{A}$:

- Conditions: open sets $U \subseteq 2^\omega$ of measure less than $\frac{1}{2}$
- Order: $V \leq U$ iff $V \supseteq U$

Properties:

- σ-linked (thus ccc)
- adds an open subset $a = \bigcup G$ of 2^ω of measure $\frac{1}{2}$ (an *amoeba real*)
- makes union of ground model null sets a null set (because $X \subseteq a + r$ for every ground model null X and every rational r)

Examples for Suslin ccc forcing 3

Amoeba forcing $\mathbb{A}$:

- Conditions: open sets $U \subseteq 2^\omega$ of measure less than $\frac{1}{2}$
- Order: $V \leq U$ iff $V \supseteq U$

Properties:

- σ-linked (thus ccc)
- adds an open subset $a = \bigcup G$ of 2^ω of measure $\frac{1}{2}$ (an *amoeba real*)
- makes union of ground model null sets a null set (because $X \subseteq a + r$ for every ground model null X and every rational r)

Coding open sets by reals we see that $\mathbb{A}$ is Suslin ccc.

Absoluteness 1

Lemma (absoluteness of maximal antichains)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc.
Then "A is a maximal antichain in $\mathbb{P}$" is a $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ statement, and therefore absolute between M and N.
If $\mathbb{P}$ is a Borel set, being a maximal antichain is in fact $\mathbf{\Pi}^1_1$.

Absoluteness 1

Lemma (absoluteness of maximal antichains)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc.
Then "A is a maximal antichain in $\mathbb{P}$" is a $\boldsymbol{\Sigma}^1_1 \cup \boldsymbol{\Pi}^1_1$ statement, and therefore absolute between M and N.
If $\mathbb{P}$ is a Borel set, being a maximal antichain is in fact $\boldsymbol{\Pi}^1_1$.

Proof: ccc: antichains are countable and coded by reals.

Absoluteness 1

Lemma (absoluteness of maximal antichains)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc. Then "A is a maximal antichain in $\mathbb{P}$" is a $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ statement, and therefore absolute between M and N. If $\mathbb{P}$ is a Borel set, being a maximal antichain is in fact $\mathbf{\Pi}^1_1$.

Proof: ccc: antichains are countable and coded by reals.
Let $A = \{x_n : n \in \omega\} \subseteq \mathbb{P}$. A is a maximal antichain iff

- $x_n \perp_{\mathbb{P}} x_m$ for all $n \neq m$ and,
- for all y, either $y \notin \mathbb{P}$ or there is n such that $y \not\perp_{\mathbb{P}} x_n$.

Absoluteness 1

Lemma (absoluteness of maximal antichains)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc.
Then "A is a maximal antichain in $\mathbb{P}$" is a $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ statement, and therefore absolute between M and N.
If $\mathbb{P}$ is a Borel set, being a maximal antichain is in fact $\mathbf{\Pi}^1_1$.

Proof: ccc: antichains are countable and coded by reals.
Let $A = \{x_n : n \in \omega\} \subseteq \mathbb{P}$. A is a maximal antichain iff

- $x_n \perp_{\mathbb{P}} x_m$ for all $n \neq m$ and,
- for all y, either $y \notin \mathbb{P}$ or there is n such that $y \not\perp_{\mathbb{P}} x_n$.

The first part is $\mathbf{\Sigma}^1_1$, while the second is $\mathbf{\Pi}^1_1$. Thus $\mathbf{\Sigma}^1_1$ absoluteness applies. □

Absoluteness 2

Corollary (downward absoluteness of genericity)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc.
If G is $\mathbb{P}^N$-generic over N, then $G \cap M$ is $\mathbb{P}^M$-generic over M.

Absoluteness 2

Corollary (downward absoluteness of genericity)

Let $M \subseteq N$ be ZFC-models. Let $\mathbb{P} \in M$ be Suslin ccc.
If G is $\mathbb{P}^N$-generic over N, then $G \cap M$ is $\mathbb{P}^M$-generic over M.

Proof: Let $A \in M$ be a maximal antichain of $\mathbb{P}$ in M.
By previous lemma: A maximal antichain of $\mathbb{P}$ in N.
Hence $G \cap A \neq \emptyset$. $\square$

Embeddability in iterations 1

Lemma (preservation of embeddability in iterations)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s. Let $\dot{\mathbb{Q}}_i$ be $\mathbb{P}_i$-names for p.o.'s such that $\mathbb{P}_1 \Vdash \dot{\mathbb{Q}}_0 \subseteq \dot{\mathbb{Q}}_1$ and all maximal antichains of $\dot{\mathbb{Q}}_0$ in $V^{\mathbb{P}_0}$ are maximal antichains of $\dot{\mathbb{Q}}_1$ in $V^{\mathbb{P}_1}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0 \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.

Embeddability in iterations 1

Lemma (preservation of embeddability in iterations)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s. Let $\dot{\mathbb{Q}}_i$ be $\mathbb{P}_i$-names for p.o.'s such that $\mathbb{P}_1 \Vdash \dot{\mathbb{Q}}_0 \subseteq \dot{\mathbb{Q}}_1$ and all maximal antichains of $\dot{\mathbb{Q}}_0$ in $V^{\mathbb{P}_0}$ are maximal antichains of $\dot{\mathbb{Q}}_1$ in $V^{\mathbb{P}_1}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0 \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.

Proof: Let A be a maximal antichain in $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0$.
Need to show: A still maximal in $\mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.
Let $(p^0, \dot{q}^0) \in \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.

Embeddability in iterations 1

Lemma (preservation of embeddability in iterations)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s. Let $\dot{\mathbb{Q}}_i$ be $\mathbb{P}_i$-names for p.o.'s such that $\mathbb{P}_1 \Vdash \dot{\mathbb{Q}}_0 \subseteq \dot{\mathbb{Q}}_1$ and all maximal antichains of $\dot{\mathbb{Q}}_0$ in $V^{\mathbb{P}_0}$ are maximal antichains of $\dot{\mathbb{Q}}_1$ in $V^{\mathbb{P}_1}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0 \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.

Proof: Let A be a maximal antichain in $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0$.
Need to show: A still maximal in $\mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.
Let $(p^0, \dot{q}^0) \in \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.
Fix $\mathbb{P}_1$-generic filter G over V containing p^0.
By assumption, $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over V.
In $V[G \cap \mathbb{P}_0]$, let

$$B = \{q \in \mathbb{Q}_0 : \exists (p, \dot{q}) \in A \text{ with } p \in G \text{ and } q = \dot{q}[G \cap \mathbb{P}_0]\}.$$

Embeddability in iterations 2

Check: B is a maximal antichain in $\mathbb{Q}_0$ in $V[G \cap \mathbb{P}_0]$!

Embeddability in iterations 2

Check: B is a maximal antichain in $\mathbb{Q}_0$ in $V[G \cap \mathbb{P}_0]$!
By assumption, B maximal in $\mathbb{Q}_1$ in $V[G]$.
Hence there is $q \in B$ compatible with $\dot{q}^0[G]$.
Let $(p, \dot{q}) \in A$ witness $q = \dot{q}[G \cap \mathbb{P}_0] \in B$.

Embeddability in iterations 2

Check: B is a maximal antichain in $\mathbb{Q}_0$ in $V[G \cap \mathbb{P}_0]$!
By assumption, B maximal in $\mathbb{Q}_1$ in $V[G]$.
Hence there is $q \in B$ compatible with $\dot{q}^0[G]$.
Let $(p, \dot{q}) \in A$ witness $q = \dot{q}[G \cap \mathbb{P}_0] \in B$.
There is $\bar{p} \in G$ forcing that $\dot{q}$ and $\dot{q}^0$ are compatible, with common extension $\dot{\bar{q}}$. Wlog $\bar{p} \leq p, p^0$. Then $(\bar{p}, \dot{\bar{q}}) \leq (p, \dot{q}), (p^0, \dot{q}^0)$. □

Embeddability in iterations 2

Check: B is a maximal antichain in $\mathbb{Q}_0$ in $V[G \cap \mathbb{P}_0]$!
By assumption, B maximal in $\mathbb{Q}_1$ in $V[G]$.
Hence there is $q \in B$ compatible with $\dot{q}^0[G]$.
Let $(p, \dot{q}) \in A$ witness $q = \dot{q}[G \cap \mathbb{P}_0] \in B$.
There is $\bar{p} \in G$ forcing that $\dot{q}$ and $\dot{q}^0$ are compatible, with common extension $\dot{\bar{q}}$. Wlog $\bar{p} \leq p, p^0$. Then $(\bar{p}, \dot{\bar{q}}) \leq (p, \dot{q}), (p^0, \dot{q}^0)$. $\square$

Corollary (embeddability of Suslin ccc forcing)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s.
Assume $\mathbb{Q}$ is a Suslin ccc forcing coded in $V^{\mathbb{P}_0}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_0}} \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_1}}$.

Embeddability in iterations 2

Check: B is a maximal antichain in $\mathbb{Q}_0$ in $V[G \cap \mathbb{P}_0]$!
By assumption, B maximal in $\mathbb{Q}_1$ in $V[G]$.
Hence there is $q \in B$ compatible with $\dot{q}^0[G]$.
Let $(p, \dot{q}) \in A$ witness $q = \dot{q}[G \cap \mathbb{P}_0] \in B$.
There is $\bar{p} \in G$ forcing that $\dot{q}$ and $\dot{q}^0$ are compatible, with common extension $\dot{\bar{q}}$. Wlog $\bar{p} \leq p, p^0$. Then $(\bar{p}, \dot{\bar{q}}) \leq (p, \dot{q}), (p^0, \dot{q}^0)$. $\square$

Corollary (embeddability of Suslin ccc forcing)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s.
Assume $\mathbb{Q}$ is a Suslin ccc forcing coded in $V^{\mathbb{P}_0}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_0}} \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_1}}$.

Proof: Immediate by previous lemma and absoluteness of maximal antichains of Suslin ccc forcing. $\square$

Finite support iteration

Let δ be an ordinal. Let $\mathbb{Q}_\alpha$, $\alpha < \delta$, be Suslin ccc, all coded in V.

Finite support iteration

Let δ be an ordinal. Let $\mathbb{Q}_\alpha$, $\alpha < \delta$, be Suslin ccc, all coded in V.

One can recursively define the *finite support iteration (fsi)* $(\mathbb{P}_\alpha : \alpha \leq \delta)$ with iterands $\mathbb{Q}_\alpha$ in the usual way, letting $\mathbb{P}_{\alpha+1}$ be the two-step iteration of $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_\alpha}}$ (the reinterpretation of $\mathbb{Q}_\alpha$ in the $\mathbb{P}_\alpha$-generic extension).

Finite support iteration

Let δ be an ordinal. Let $\mathbb{Q}_\alpha$, $\alpha < \delta$, be Suslin ccc, all coded in V.

One can recursively define the *finite support iteration (fsi)* $(\mathbb{P}_\alpha : \alpha \leq \delta)$ with iterands $\mathbb{Q}_\alpha$ in the usual way, letting $\mathbb{P}_{\alpha+1}$ be the two-step iteration of $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_\alpha}}$ (the reinterpretation of $\mathbb{Q}_\alpha$ in the $\mathbb{P}_\alpha$-generic extension).

We will also look at fragments of this iteration.
By the absoluteness properties described above, all these fragments will completely embed into the whole iteration in a canonical way.

Fragments of the iteration

Fix $X \subseteq \delta$.
By recursion on $\alpha \leq \delta$, define the p.o. $\mathbb{P}_{X \cap \alpha}$:

- $\mathbb{P}_{X \cap 0} = \{1\}$

Fragments of the iteration

Fix $X \subseteq \delta$.
By recursion on $\alpha \leq \delta$, define the p.o. $\mathbb{P}_{X\cap\alpha}$:

- $\mathbb{P}_{X\cap 0} = \{1\}$
- $\mathbb{P}_{X\cap(\alpha+1)} = \begin{cases} \mathbb{P}_{X\cap\alpha} & \text{if } \alpha \notin X \\ \mathbb{P}_{X\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{X\cap\alpha}}} & \text{if } \alpha \in X \end{cases}$

Fragments of the iteration

Fix $X \subseteq \delta$.
By recursion on $\alpha \leq \delta$, define the p.o. $\mathbb{P}_{X\cap\alpha}$:

- $\mathbb{P}_{X\cap 0} = \{1\}$
- $\mathbb{P}_{X\cap(\alpha+1)} = \begin{cases} \mathbb{P}_{X\cap\alpha} & \text{if } \alpha \notin X \\ \mathbb{P}_{X\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{X\cap\alpha}}} & \text{if } \alpha \in X \end{cases}$
- For limit γ, $\mathbb{P}_{X\cap\gamma} = \lim\operatorname{dir}_{\alpha<\gamma}\mathbb{P}_{X\cap\alpha}$

Fragments of the iteration

Fix $X \subseteq \delta$.
By recursion on $\alpha \leq \delta$, define the p.o. $\mathbb{P}_{X \cap \alpha}$:

- $\mathbb{P}_{X \cap 0} = \{1\}$
- $\mathbb{P}_{X \cap (\alpha+1)} = \begin{cases} \mathbb{P}_{X \cap \alpha} & \text{if } \alpha \notin X \\ \mathbb{P}_{X \cap \alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{X \cap \alpha}}} & \text{if } \alpha \in X \end{cases}$
- For limit γ, $\mathbb{P}_{X \cap \gamma} = \lim \mathrm{dir}_{\alpha < \gamma} \mathbb{P}_{X \cap \alpha}$

Clearly, for $X = \delta$ one obtains the standard fsi $(\mathbb{P}_\alpha : \alpha \leq \delta)$ mentioned above.

Embeddability of fragments 1

Lemma (embeddability of fragments)

Assume $X \subseteq Y \subseteq \delta$. *Then* $\mathbb{P}_X \lessdot \mathbb{P}_Y$.

Embeddability of fragments 1

Lemma (embeddability of fragments)

Assume $X \subseteq Y \subseteq \delta$. *Then* $\mathbb{P}_X \lessdot \mathbb{P}_Y$.

Proof: Prove by induction on $\alpha \leq \delta$ that $\mathbb{P}_{X\cap\alpha} \lessdot \mathbb{P}_{Y\cap\alpha}$.

Basic step: trivial.

Embeddability of fragments 1

Lemma (embeddability of fragments)

Assume $X \subseteq Y \subseteq \delta$. Then $\mathbb{P}_X \lessdot \mathbb{P}_Y$.

Proof: Prove by induction on $\alpha \leq \delta$ that $\mathbb{P}_{X \cap \alpha} \lessdot \mathbb{P}_{Y \cap \alpha}$.

Basic step: trivial.

Successor step: let $\beta = \alpha + 1$.
If $\alpha \notin X$,

$$\mathbb{P}_{X \cap \beta} = \mathbb{P}_{X \cap \alpha} \lessdot \mathbb{P}_{Y \cap \alpha} \lessdot \mathbb{P}_{Y \cap \beta}$$

by definition and induction hypothesis.

Embeddability of fragments 2

So assume $\alpha \in X$. Recall:

Corollary (embeddability of Suslin ccc forcing)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s.
Assume $\mathbb{Q}$ is a Suslin ccc forcing coded in $V^{\mathbb{P}_0}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_0}} \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}^{V^{\mathbb{P}_1}}$.

By induction hypothesis and embeddability of Suslin ccc forcing,

$$\mathbb{P}_{X\cap\beta} = \mathbb{P}_{X\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{X\cap\alpha}}} \lessdot \mathbb{P}_{Y\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{Y\cap\alpha}}} = \mathbb{P}_{Y\cap\beta}$$

Embeddability of fragments 2

So assume $\alpha \in X$. By induction hypothesis and embeddability of Suslin ccc forcing,

$$\mathbb{P}_{X\cap\beta} = \mathbb{P}_{X\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{X\cap\alpha}}} <\circ\, \mathbb{P}_{Y\cap\alpha} \star \dot{\mathbb{Q}}_\alpha^{V^{\mathbb{P}_{Y\cap\alpha}}} = \mathbb{P}_{Y\cap\beta}$$

Limit step: exercise! □

Localization 1

Lemma (localization)

Let $\alpha \leq \delta$.

(i) *Let $p \in \mathbb{P}_\alpha$.*
Then there is $X \subseteq \alpha$ countable such that $p \in \mathbb{P}_X$.

(ii) *Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for a real.*
Then there is $X \subseteq \alpha$ countable such that $\dot{f}$ is a $\mathbb{P}_X$-name for a real.

Localization 1

Lemma (localization)

Let $\alpha \leq \delta$.

(i) *Let $p \in \mathbb{P}_\alpha$.*
Then there is $X \subseteq \alpha$ countable such that $p \in \mathbb{P}_X$.

(ii) *Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for a real.*
Then there is $X \subseteq \alpha$ countable such that $\dot{f}$ is a $\mathbb{P}_X$-name for a real.

Proof: Simultaneous induction on $\alpha \leq \delta$.

Basic step: trivial.

Localization 2

Successor step: let $\beta = \alpha + 1$.
(i) Let $(p, \dot{q}) \in \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha = \mathbb{P}_\beta$.
By induction hypothesis for (i) and (ii): there are countable X_0 and X_1 such that $p \in \mathbb{P}_{X_0}$ and $\dot{q}$ is a $\mathbb{P}_{X_1}$-name.
Let $X = X_0 \cup X_1 \cup \{\alpha\}$. Then $(p, \dot{q}) \in \mathbb{P}_X$.

Localization 2

Successor step: let $\beta = \alpha + 1$.
(i) Let $(p, \dot{q}) \in \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha = \mathbb{P}_\beta$.
By induction hypothesis for (i) and (ii): there are countable X_0 and X_1 such that $p \in \mathbb{P}_{X_0}$ and $\dot{q}$ is a $\mathbb{P}_{X_1}$-name.
Let $X = X_0 \cup X_1 \cup \{\alpha\}$. Then $(p, \dot{q}) \in \mathbb{P}_X$.
(ii) Let $\dot{f}$ be a $\mathbb{P}_\beta$-name for a real.
There a countable maximal antichains $\{p_n^m : m \in \omega\} \subseteq \mathbb{P}_\beta$ and numbers $\{k_n^m : m \in \omega\}$, such that $p_n^m \Vdash \dot{f}(n) = k_n^m$.
By (i): there are countable X_n^m such that $p_n^m \in \mathbb{P}_{X_n^m}$.
Let $X = \bigcup_{n,m} X_n^m$.
Since $\dot{f}$ is completely decided by p_n^m and k_n^m, it is $\mathbb{P}_X$-name.

Localization 2

Successor step: let $\beta = \alpha + 1$.
(i) Let $(p, \dot{q}) \in \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha = \mathbb{P}_\beta$.
By induction hypothesis for (i) and (ii): there are countable X_0 and X_1 such that $p \in \mathbb{P}_{X_0}$ and $\dot{q}$ is a $\mathbb{P}_{X_1}$-name.
Let $X = X_0 \cup X_1 \cup \{\alpha\}$. Then $(p, \dot{q}) \in \mathbb{P}_X$.
(ii) Let $\dot{f}$ be a $\mathbb{P}_\beta$-name for a real.
There a countable maximal antichains $\{p_n^m : m \in \omega\} \subseteq \mathbb{P}_\beta$ and numbers $\{k_n^m : m \in \omega\}$, such that $p_n^m \Vdash \dot{f}(n) = k_n^m$.
By (i): there are countable X_n^m such that $p_n^m \in \mathbb{P}_{X_n^m}$.
Let $X = \bigcup_{n,m} X_n^m$.
Since $\dot{f}$ is completely decided by p_n^m and k_n^m, it is $\mathbb{P}_X$-name.

Limit step: (i) trivial. (ii) follows from (i) as above. $\square$

Direct limit 1

Corollary (representation as direct limit)

Let $\mathcal{X} \subseteq \mathcal{P}(\delta)$ be a directed family of sets such that for every countable $Y \subseteq \delta$ there is $X \in \mathcal{X}$ with $Y \subseteq X$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_{X \in \mathcal{X}} \mathbb{P}_X$.

Direct limit 1

Corollary (representation as direct limit)

Let $\mathcal{X} \subseteq \mathcal{P}(\delta)$ be a directed family of sets such that for every countable $Y \subseteq \delta$ there is $X \in \mathcal{X}$ with $Y \subseteq X$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_{X \in \mathcal{X}} \mathbb{P}_X$.

Proof:
By embeddability of fragments, the direct limit is a subset of $\mathbb{P}_\delta$.
By localization, then, the two sets are actually equal. □

Direct limit 1

Corollary (representation as direct limit)

Let $\mathcal{X} \subseteq \mathcal{P}(\delta)$ be a directed family of sets such that for every countable $Y \subseteq \delta$ there is $X \in \mathcal{X}$ with $Y \subseteq X$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_{X \in \mathcal{X}} \mathbb{P}_X$.

Proof:
By embeddability of fragments, the direct limit is a subset of $\mathbb{P}_\delta$.
By localization, then, the two sets are actually equal. $\square$

Corollary

$\mathbb{P}_\delta = \lim \mathrm{dir}\{\mathbb{P}_X : X \subseteq \delta \textit{ is countable}\}$.

Direct limit 1

Corollary (representation as direct limit)

Let $\mathcal{X} \subseteq \mathcal{P}(\delta)$ be a directed family of sets such that for every countable $Y \subseteq \delta$ there is $X \in \mathcal{X}$ with $Y \subseteq X$.
Then $\mathbb{P}_\delta = \lim\mathrm{dir}_{X \in \mathcal{X}} \mathbb{P}_X$.

Proof:
By embeddability of fragments, the direct limit is a subset of $\mathbb{P}_\delta$.
By localization, then, the two sets are actually equal. □

Corollary

$\mathbb{P}_\delta = \lim\mathrm{dir}\{\mathbb{P}_X : X \subseteq \delta \textit{ is countable}\}$.

Question

What can we say about the direct limit of finite fragments of Suslin ccc iterations? E.g., for Hechler forcing.

Direct limit 2

Lemma

Assume $\mathbb{P}$ is Suslin ccc, and $\mathbb{P}_\delta$ is an iteration of Suslin ccc forcing.
Consider $\mathbb{P} \star \dot{\mathbb{P}}_\delta$.
No new real of $V^{\mathbb{P}} \setminus V$ belongs to $V^{\mathbb{P}_\delta}$ (in $V^{\mathbb{P} \star \dot{\mathbb{P}}_\delta}$).

Direct limit 2

Lemma

*Assume $\mathbb{P}$ is Suslin ccc, and $\mathbb{P}_\delta$ is an iteration of Suslin ccc forcing. Consider $\mathbb{P} \star \dot{\mathbb{P}}_\delta$.
No new real of $V^{\mathbb{P}} \setminus V$ belongs to $V^{\mathbb{P}_\delta}$ (in $V^{\mathbb{P}\star\dot{\mathbb{P}}_\delta}$).*

Warning: This is not true for iterations of forcing notions in general. For example, if s_0 is Sacks generic over V, and s_1 is Sacks generic over $V[s_0]$, then $s_0 \in V[s_1]$.

Direct limit 2

Lemma

Assume $\mathbb{P}$ is Suslin ccc, and $\mathbb{P}_\delta$ is an iteration of Suslin ccc forcing.
Consider $\mathbb{P} \star \dot{\mathbb{P}}_\delta$.
No new real of $V^{\mathbb{P}} \setminus V$ belongs to $V^{\mathbb{P}_\delta}$ (in $V^{\mathbb{P}\star\dot{\mathbb{P}}_\delta}$).

Corollary (representation as ω_1-stage direct limit)

Let δ be uncountable. Let X_α, $\alpha < \omega_1$, be a strictly increasing sequence of subsets of δ with $\delta = \bigcup_\alpha X_\alpha$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_\alpha \mathbb{P}_{X_\alpha}$. Furthermore,

(i) $\omega^\omega \cap V^{\mathbb{P}_\delta} = \bigcup_\alpha (\omega^\omega \cap V^{\mathbb{P}_{X_\alpha}})$

(ii) $\omega^\omega \cap (V^{\mathbb{P}_{X_{\alpha+1}}} \setminus V^{\mathbb{P}_{X_\alpha}}) \neq \emptyset$ *for* $\alpha < \omega_1$

Direct limit 2

Lemma

Assume $\mathbb{P}$ is Suslin ccc, and $\mathbb{P}_\delta$ is an iteration of Suslin ccc forcing.
Consider $\mathbb{P} \star \dot{\mathbb{P}}_\delta$.
No new real of $V^{\mathbb{P}} \setminus V$ belongs to $V^{\mathbb{P}_\delta}$ (in $V^{\mathbb{P}\star\dot{\mathbb{P}}_\delta}$).

Corollary (representation as ω_1-stage direct limit)

Let δ be uncountable. Let X_α, $\alpha < \omega_1$, be a strictly increasing sequence of subsets of δ with $\delta = \bigcup_\alpha X_\alpha$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_\alpha \mathbb{P}_{X_\alpha}$. Furthermore,

(i) $\omega^\omega \cap V^{\mathbb{P}_\delta} = \bigcup_\alpha (\omega^\omega \cap V^{\mathbb{P}_{X_\alpha}})$

(ii) $\omega^\omega \cap (V^{\mathbb{P}_{X_{\alpha+1}}} \setminus V^{\mathbb{P}_{X_\alpha}}) \neq \emptyset$ for $\alpha < \omega_1$

Proof: first part: representation as direct limit.
second part: (i) localization. (ii) apply lemma above. □

Cardinal invariants of the continuum 1

For our applications, we need some of the basic *cardinal invariants of the continuum*.

Cardinal invariants of the continuum 1

For our applications, we need some of the basic *cardinal invariants of the continuum*.

For $f, g \in \omega^\omega$:

$f \leq^* g$ (g *eventually dominates* f)

$\iff f(n) \leq g(n)$ for all but finitely many n

Cardinal invariants of the continuum 1

For our applications, we need some of the basic *cardinal invariants of the continuum*.

For $f, g \in \omega^\omega$:

$$f \leq^* g \qquad (g \textit{ eventually dominates } f)$$
$$\iff f(n) \leq g(n) \text{ for all but finitely many } n$$

$\mathfrak{b} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is unbounded in } (\omega^\omega, \leq^*)\}$, the *bounding number*.

$\mathfrak{d} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is cofinal in } (\omega^\omega, \leq^*)\}$, the *dominating number*.

Cardinal invariants of the continuum 2

For $A, B \subseteq \omega$:

$$A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite}$$

Cardinal invariants of the continuum 2

For $A, B \subseteq \omega$:

$$A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite}$$

For $A, B \in [\omega]^\omega$:

$$A \textit{ splits } B \iff |A \cap B| = |B \setminus A| = \aleph_0$$

Cardinal invariants of the continuum 2

For $A, B \subseteq \omega$:

$$A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite}$$

For $A, B \in [\omega]^\omega$:

$$A \textit{ splits } B \iff |A \cap B| = |B \setminus A| = \aleph_0$$

$\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if every member of $[\omega]^\omega$ is split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq [\omega]^\omega$ is *unsplit* (or *unreaped*) if no member of $[\omega]^\omega$ splits all members of $\mathcal{F}$. I.e. $\forall A \in [\omega]^\omega \, \exists B \in \mathcal{F} \, (|A \cap B| < \aleph_0$ or $B \subseteq^* A)$

Cardinal invariants of the continuum 2

For $A, B \subseteq \omega$:

$$A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite}$$

For $A, B \in [\omega]^\omega$:

$$A \textit{ splits } B \iff |A \cap B| = |B \setminus A| = \aleph_0$$

$\mathcal{F} \subseteq [\omega]^\omega$ is *splitting* if every member of $[\omega]^\omega$ is split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq [\omega]^\omega$ is *unsplit* (or *unreaped*) if no member of $[\omega]^\omega$ splits all members of $\mathcal{F}$.

$\mathfrak{s} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is splitting}\}$, the *splitting number*.
$\mathfrak{r} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is unsplit}\}$, the *reaping number*.

Cardinal invariants of the continuum 3

$\mathcal{D} \subseteq [\omega]^\omega$ *dense*: $\forall A \in [\omega]^\omega \ \ \exists B \in \mathcal{D} \ \ (B \subseteq^* A)$

Cardinal invariants of the continuum 3

$\mathcal{D} \subseteq [\omega]^\omega$ *dense*: $\forall A \in [\omega]^\omega \ \exists B \in \mathcal{D} \ (B \subseteq^* A)$

$\mathcal{D} \subseteq [\omega]^\omega$ *open*: $\forall A \in \mathcal{D} \ \forall B \subseteq^* A \ (B \in \mathcal{D})$

Cardinal invariants of the continuum 3

$\mathcal{D} \subseteq [\omega]^\omega$ *dense*: $\forall A \in [\omega]^\omega \ \exists B \in \mathcal{D} \ (B \subseteq^* A)$
$\mathcal{D} \subseteq [\omega]^\omega$ *open*: $\forall A \in \mathcal{D} \ \forall B \subseteq^* A \ (B \in \mathcal{D})$

A family $\mathcal{D} \subseteq [\omega]^\omega$ is *groupwise dense* if

- $\mathcal{D}$ is open
- given a partition $(I_n : n \in \omega)$ of ω into intervals, there is $B \in [\omega]^\omega$ such that $\bigcup_{n \in B} I_n \in \mathcal{D}$
(this implies, in particular, that $\mathcal{D}$ is dense)

Cardinal invariants of the continuum 3

$\mathcal{D} \subseteq [\omega]^\omega$ *dense*: $\forall A \in [\omega]^\omega \; \exists B \in \mathcal{D} \; (B \subseteq^* A)$
$\mathcal{D} \subseteq [\omega]^\omega$ *open*: $\forall A \in \mathcal{D} \; \forall B \subseteq^* A \; (B \in \mathcal{D})$

A family $\mathcal{D} \subseteq [\omega]^\omega$ is *groupwise dense* if

- $\mathcal{D}$ is open
- given a partition $(I_n : n \in \omega)$ of ω into intervals, there is $B \in [\omega]^\omega$ such that $\bigcup_{n \in B} I_n \in \mathcal{D}$
(this implies, in particular, that $\mathcal{D}$ is dense)

$\mathfrak{h} := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ open dense and } \bigcap \mathfrak{D} = \emptyset\}$
the *distributivity number*.
$\mathfrak{g} := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ groupwise dense and } \bigcap \mathfrak{D} = \emptyset\}$
the *groupwise density number*.

Cardinal invariants of the continuum 4

$\mathcal{I}$ ideal on the reals.

$\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I}$ and $\bigcup \mathcal{F} \notin \mathcal{I}\}$, the *additivity* of $\mathcal{I}$.
$\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I}$ is a basis$\}$, the *cofinality* of $\mathcal{I}$.

Basis: $\mathcal{F} \subseteq \mathcal{I}$ such every member of $\mathcal{I}$ is contained in some member of $\mathcal{F}$.

Cardinal invariants of the continuum 4

$\mathcal{I}$ ideal on the reals.

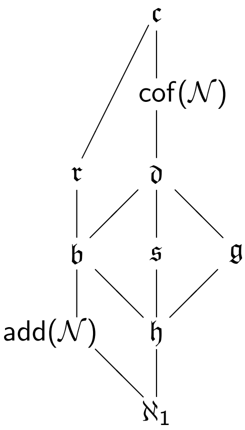
$\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} \notin \mathcal{I}\}$, the *additivity* of $\mathcal{I}$.
$\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ is a basis}\}$, the *cofinality* of $\mathcal{I}$.

Basis: $\mathcal{F} \subseteq \mathcal{I}$ such every member of $\mathcal{I}$ is contained in some member of $\mathcal{F}$.

Theorem

(i) $\mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}, \mathfrak{g}\}$ *and* $\mathfrak{g} \leq \mathfrak{d}$
(ii) $\mathfrak{b} \leq \mathfrak{d}$
(iii) $\mathfrak{b} \leq \mathfrak{r}$ *and dually* $\mathfrak{s} \leq \mathfrak{d}$
(iv) $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$ *and dually* $\mathfrak{d} \leq \mathrm{cof}(\mathcal{N})$ *for the null ideal* $\mathcal{N}$

ZFC-inequalities: a diagram



First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 1

Theorem

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then, in the $\mathbb{P}_\lambda$-extension, $\mathfrak{g} = \aleph_1$.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 1

Theorem

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then, in the $\mathbb{P}_\lambda$-extension, $\mathfrak{g} = \aleph_1$.

Corollary

Let $\mathbb{D}_\lambda$ be the fsi of Hechler forcing $\mathbb{D}$.
In the $\mathbb{D}_\lambda$-extension, $\mathfrak{b} = \mathfrak{d} = \lambda$ while $\mathfrak{g} = \aleph_1$.
In particular, $\mathfrak{g} < \mathfrak{b}$ is consistent.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 1

Theorem

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then, in the $\mathbb{P}_\lambda$-extension, $\mathfrak{g} = \aleph_1$.

Corollary

Let $\mathbb{D}_\lambda$ be the fsi of Hechler forcing $\mathbb{D}$.
In the $\mathbb{D}_\lambda$-extension, $\mathfrak{b} = \mathfrak{d} = \lambda$ while $\mathfrak{g} = \aleph_1$.
In particular, $\mathfrak{g} < \mathfrak{b}$ is consistent.

Proof: $\mathfrak{b} = \mathfrak{d} = \lambda$ because we add a λ-*scale*
(a well-ordered dominating family of size λ).
$\mathfrak{g} = \aleph_1$ follows from Theorem. $\square$

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 1

Theorem

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then, in the $\mathbb{P}_\lambda$-extension, $\mathfrak{g} = \aleph_1$.

Corollary

Let $\mathbb{D}_\lambda$ be the fsi of Hechler forcing $\mathbb{D}$.
In the $\mathbb{D}_\lambda$-extension, $\mathfrak{b} = \mathfrak{d} = \lambda$ while $\mathfrak{g} = \aleph_1$.
In particular, $\mathfrak{g} < \mathfrak{b}$ is consistent.

Corollary

Let $\mathbb{A}_\lambda$ be the fsi of amoeba forcing $\mathbb{A}$.
In the $\mathbb{A}_\lambda$-extension, $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \lambda$ while $\mathfrak{g} = \aleph_1$.
In particular, $\mathfrak{g} < \text{add}(\mathcal{N})$ is consistent.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 2

Theorem follows from:

Corollary (representation as ω_1-stage direct limit)

Let δ be uncountable. Let X_α, $\alpha < \omega_1$ be a strictly increasing sequence of subsets of δ with $\delta = \bigcup_\alpha X_\alpha$.
Then $\mathbb{P}_\delta = \lim \mathrm{dir}_\alpha \mathbb{P}_{X_\alpha}$. Furthermore,

(i) $\omega^\omega \cap V^{\mathbb{P}_\delta} = \bigcup_\alpha (\omega^\omega \cap V^{\mathbb{P}_{X_\alpha}})$

(ii) $\omega^\omega \cap (V^{\mathbb{P}_{X_{\alpha+1}}} \setminus V^{\mathbb{P}_{X_\alpha}}) \neq \emptyset$ *for* $\alpha < \omega_1$

and the following lemma:

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 3

Lemma

Let κ be an uncountable cardinal. Assume there is an increasing chain of ZFC-models V_α, $\alpha < \kappa$, such that

(i) $\omega^\omega \cap V = \bigcup_{\alpha<\kappa}(\omega^\omega \cap V_\alpha)$

(ii) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$ *for all* $\alpha < \kappa$.

Then $\mathfrak{g} \leq \kappa$.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 3

Lemma

Let κ be an uncountable cardinal. Assume there is an increasing chain of ZFC-models V_α, $\alpha < \kappa$, such that

(i) $\omega^\omega \cap V = \bigcup_{\alpha<\kappa}(\omega^\omega \cap V_\alpha)$

(ii) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$ *for all* $\alpha < \kappa$.

Then $\mathfrak{g} \leq \kappa$.

Proof: Let

$$\mathcal{D}_\alpha = \{X \in [\omega]^\omega : X \text{ has no almost subset in } V_\alpha\}$$

(i): intersection of $\mathcal{D}_\alpha$ is empty.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 3

Lemma

Let κ be an uncountable cardinal. Assume there is an increasing chain of ZFC-models V_α, $\alpha < \kappa$, such that

(i) $\omega^\omega \cap V = \bigcup_{\alpha<\kappa}(\omega^\omega \cap V_\alpha)$

(ii) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$ *for all* $\alpha < \kappa$.

Then $\mathfrak{g} \leq \kappa$.

Proof: Let

$$\mathcal{D}_\alpha = \{X \in [\omega]^\omega : X \text{ has no almost subset in } V_\alpha\}$$

(i): intersection of $\mathcal{D}_\alpha$ is empty.
Check the $\mathcal{D}_\alpha$ are groupwise dense.
Obviously, they are open.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 4

Let $\mathcal{I} = (I_n : n \in \omega)$ be a partition of ω into intervals.
(i): there is $\beta \geq \alpha$ such that $\mathcal{I} \in V_\beta$.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 4

Let $\mathcal{I} = (I_n : n \in \omega)$ be a partition of ω into intervals.
(i): there is $\beta \geq \alpha$ such that $\mathcal{I} \in V_\beta$.
Let $\mathcal{A} \in V_\beta$ be a mad family which contains a perfect a.d. family $\mathcal{B}$.
(ii): $\mathcal{B}$ has new branch A in $V_{\beta+1}$.
A almost disjoint from $\mathcal{A}$. Let $C = \bigcup_{n \in A} I_n$.

Claim: $C \in \mathcal{D}_\beta$ and thus $C \in \mathcal{D}_\alpha$ as well.

First application: $\mathfrak{b}$ versus $\mathfrak{g}$ 4

Let $\mathcal{I} = (I_n : n \in \omega)$ be a partition of ω into intervals.
(i): there is $\beta \geq \alpha$ such that $\mathcal{I} \in V_\beta$.
Let $\mathcal{A} \in V_\beta$ be a mad family which contains a perfect a.d. family $\mathcal{B}$.
(ii): $\mathcal{B}$ has new branch A in $V_{\beta+1}$.
A almost disjoint from $\mathcal{A}$. Let $C = \bigcup_{n \in A} I_n$.

Claim: $C \in \mathcal{D}_\beta$ and thus $C \in \mathcal{D}_\alpha$ as well.

Suppose C has an almost subset $D \in V_\beta$.
Let $E = \{n : I_n \cap D \neq \emptyset\}$.
Clearly $E \subseteq^* A$ so that E is almost disjoint from $\mathcal{A}$.
On the other hand, E belongs to V_β because both D and $\mathcal{I}$ do.
This contradicts the maximality of $\mathcal{A}$. □

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Judah-Shelah)

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then the ground model reals form a splitting family in the $\mathbb{P}_\lambda$-extension.

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Judah-Shelah)

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then the ground model reals form a splitting family in the $\mathbb{P}_\lambda$-extension.

Corollary (Judah-Shelah)

$\mathfrak{s} < \mathfrak{b}$ *is consistent. Even* $\text{add}(\mathcal{N}) < \mathfrak{b}$ *is consistent.*

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Judah-Shelah)

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then the ground model reals form a splitting family in the $\mathbb{P}_\lambda$-extension.

Corollary (Judah-Shelah)

$\mathfrak{s} < \mathfrak{b}$ *is consistent. Even* $\text{add}(\mathcal{N}) < \mathfrak{b}$ *is consistent.*

Proof: Use again iteration of $\mathbb{D}$ (Hechler) or $\mathbb{A}$ (amoeba). □

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Judah-Shelah)

Let λ be regular uncountable. Let $\mathbb{P}_\lambda$ be an fsi of Suslin ccc forcing.
Then the ground model reals form a splitting family in the $\mathbb{P}_\lambda$-extension.

Corollary (Judah-Shelah)

$\mathfrak{s} < \mathfrak{b}$ *is consistent. Even* $\text{add}(\mathcal{N}) < \mathfrak{b}$ *is consistent.*

Proof: Use again iteration of $\mathbb{D}$ (Hechler) or $\mathbb{A}$ (amoeba). □

Remark: $CON(\mathfrak{s} < \mathfrak{b})$ was first shown by Baumgartner-Dordal using the same model but a different argument.

Absoluteness for non-definable forcing?

We investigate the problem to which extent the embeddability results and iteration techniques of lecture 1 can be generalized to the non-definable context.

Absoluteness for non-definable forcing?

We investigate the problem to which extent the embeddability results and iteration techniques of lecture 1 can be generalized to the non-definable context.

Since absoluteness of maximal antichains usually fails badly for non-ccc p.o.'s, we stay in the realm of ccc forcing.
Relatively simple non-definable ccc forcing notions can be associated naturally with ultrafilters on ω.

Mathias forcing

Let $\mathcal{F}$ be a filter on ω.

Mathias forcing with $\mathcal{F}$, $\mathbb{M}_{\mathcal{F}}$:

- Conditions: pairs (s, A) such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, and $\max s < \min A$
- Order: $(t, B) \leq (s, A)$ if $t \supseteq s$, $t \setminus s \subseteq A$, and $B \subseteq A$

Mathias forcing

Let $\mathcal{F}$ be a filter on ω.
Mathias forcing with $\mathcal{F}$, $\mathbb{M}_{\mathcal{F}}$:

- Conditions: pairs (s, A) such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, and $\max s < \min A$
- Order: $(t, B) \leq (s, A)$ if $t \supseteq s$, $t \setminus s \subseteq A$, and $B \subseteq A$

Properties:

- σ-centered

Mathias forcing

Let $\mathcal{F}$ be a filter on ω.
Mathias forcing with $\mathcal{F}$, $\mathbb{M}_{\mathcal{F}}$:

- Conditions: pairs (s, A) such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, and $\max s < \min A$
- Order: $(t, B) \leq (s, A)$ if $t \supseteq s$, $t \setminus s \subseteq A$, and $B \subseteq A$

Properties:

- σ-centered
- adds a generic *Mathias real*

$$m = \bigcup \{s : \text{ there is } A \in \mathcal{F} \text{ such that } (s, A) \in G\}$$

Mathias forcing

Let $\mathcal{F}$ be a filter on ω.
Mathias forcing with $\mathcal{F}$, $\mathbb{M}_{\mathcal{F}}$:

- Conditions: pairs (s, A) such that $s \in [\omega]^{<\omega}$, $A \in \mathcal{F}$, and $\max s < \min A$
- Order: $(t, B) \leq (s, A) \quad$ if $t \supseteq s$, $t \setminus s \subseteq A$, and $B \subseteq A$

Properties:

- σ-centered
- adds a generic *Mathias real*

$$m = \bigcup \{s : \text{ there is } A \in \mathcal{F} \text{ such that } (s, A) \in G\}$$

- m is a *pseudointersection* of the filter $\mathcal{F}$ ($m \subseteq^* A$ for all $A \in \mathcal{F}$)

Laver forcing

Laver forcing with $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$:

- Conditions: trees $T \subseteq \omega^{<\omega}$ such that: for all $s \in T$ with $\mathrm{stem}(T) \subseteq s$, $\mathrm{succ}_T(s) = \{n : s\hat{\ }n \in T\} \in \mathcal{F}$.
- Order: inclusion

Laver forcing

Laver forcing with $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$:

- Conditions: trees $T \subseteq \omega^{<\omega}$ such that: for all $s \in T$ with $\mathrm{stem}(T) \subseteq s$, $\mathrm{succ}_T(s) = \{n : s\hat{\ }n \in T\} \in \mathcal{F}$.
- Order: inclusion

Properties:

- σ-centered

Laver forcing

Laver forcing with $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$:

- Conditions: trees $T \subseteq \omega^{<\omega}$ such that: for all $s \in T$ with $\mathrm{stem}(T) \subseteq s$, $\mathrm{succ}_T(s) = \{n : s\hat{\ }n \in T\} \in \mathcal{F}$.
- Order: inclusion

Properties:

- σ-centered
- adds a generic *Laver real*

$$\ell = \bigcup\{\mathrm{stem}(T) : T \in G\}$$

Laver forcing

Laver forcing with $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$:

- Conditions: trees $T \subseteq \omega^{<\omega}$ such that: for all $s \in T$ with $\mathrm{stem}(T) \subseteq s$, $\mathrm{succ}_T(s) = \{n : s\hat{\ }n \in T\} \in \mathcal{F}$.
- Order: inclusion

Properties:

- σ-centered
- adds a generic *Laver real*

$$\ell = \bigcup \{\mathrm{stem}(T) : T \in G\}$$

- ℓ is a dominating real

Laver forcing

Laver forcing with $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$:

- Conditions: trees $T \subseteq \omega^{<\omega}$ such that: for all $s \in T$ with $\mathrm{stem}(T) \subseteq s$, $\mathrm{succ}_T(s) = \{n : s\hat{\ }n \in T\} \in \mathcal{F}$.
- Order: inclusion

Properties:

- σ-centered
- adds a generic *Laver real*

$$\ell = \bigcup \{\mathrm{stem}(T) : T \in G\}$$

- ℓ is a dominating real
- $\mathrm{ran}(\ell)$ is a pseudointersection of $\mathcal{F}$

Absoluteness for Mathias or Laver forcing?

Assume we have models $M \subseteq N$, and filters $\mathcal{F} \in M$ and $\mathcal{G} \in N$ extending $\mathcal{F}$.

Absoluteness for Mathias or Laver forcing?

Assume we have models $M \subseteq N$, and filters $\mathcal{F} \in M$ and $\mathcal{G} \in N$ extending $\mathcal{F}$.

Under which circumstances is every maximal antichain $A \subseteq \mathbb{M}_{\mathcal{F}}$ in M still a maximal antichain of $\mathbb{M}_{\mathcal{G}}$ is N? What about $\mathbb{L}_{\mathcal{F}}$ and $\mathbb{L}_{\mathcal{G}}$?

Absoluteness for Mathias or Laver forcing?

Assume we have models $M \subseteq N$, and filters $\mathcal{F} \in M$ and $\mathcal{G} \in N$ extending $\mathcal{F}$.

Under which circumstances is every maximal antichain $A \subseteq \mathbb{M}_{\mathcal{F}}$ in M still a maximal antichain of $\mathbb{M}_{\mathcal{G}}$ is N? What about $\mathbb{L}_{\mathcal{F}}$ and $\mathbb{L}_{\mathcal{G}}$? This is trivially true if $\mathcal{G} = \mathcal{F}$, but the situation we are interested in is when $\mathcal{G}$ properly extends $\mathcal{F}$.

The answer is easier for Laver forcing:

Absoluteness for Laver forcing 1

Lemma (preservation of maximal antichains)

The following are equivalent:

(i) *every $\mathcal{F}$-positive set in M is still $\mathcal{G}$-positive in N*

(ii) *every maximal antichain of $\mathbb{L}_{\mathcal{F}}$ in M is still a maximal antichain of $\mathbb{L}_{\mathcal{G}}$ in N*

Absoluteness for Laver forcing 1

Lemma (preservation of maximal antichains)

The following are equivalent:

(i) *every $\mathcal{F}$-positive set in M is still $\mathcal{G}$-positive in N*

(ii) *every maximal antichain of $\mathbb{L}_{\mathcal{F}}$ in M is still a maximal antichain of $\mathbb{L}_{\mathcal{G}}$ in N*

Proof: Backwards direction: easy!

Absoluteness for Laver forcing 1

Lemma (preservation of maximal antichains)

The following are equivalent:

(i) *every $\mathcal{F}$-positive set in M is still $\mathcal{G}$-positive in N*

(ii) *every maximal antichain of $\mathbb{L}_{\mathcal{F}}$ in M is still a maximal antichain of $\mathbb{L}_{\mathcal{G}}$ in N*

Proof: Backwards direction: easy!
Assume $X \in M$ is $\mathcal{F}$-positive, but $\omega \setminus X \in \mathcal{G}$. Then:

$$D = \{T \in \mathbb{L}_{\mathcal{F}} : \mathrm{stem}(T)(|\mathrm{stem}(T)| - 1) \in X\}$$

dense in $\mathbb{L}_{\mathcal{F}}$.
Yet: $S = (\omega \setminus X)^{<\omega} \in \mathbb{L}_{\mathcal{G}}$ is incompatible with every element of D.
Thus no maximal antichain $A \subseteq D$ of M survives.

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if

Absoluteness for Laver forcing 2

Forwards direction: rank argument!
Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if
  - there is no $\beta < \alpha$ with $\mathrm{rank}(s) = \beta$, and

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if
  - there is no $\beta < \alpha$ with $\mathrm{rank}(s) = \beta$, and
  - $\{n : \mathrm{rank}(s\hat{\ }n) < \alpha\}$ is $\mathcal{F}$-positive.

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if
  - there is no $\beta < \alpha$ with $\mathrm{rank}(s) = \beta$, and
  - $\{n : \mathrm{rank}(s\hat{\ }n) < \alpha\}$ is $\mathcal{F}$-positive.

Claim: for every $s \in \omega^{<\omega}$, $\mathrm{rank}(s)$ defined (thus $< \omega_1$).

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if
  - there is no $\beta < \alpha$ with $\mathrm{rank}(s) = \beta$, and
  - $\{n : \mathrm{rank}(s\hat{\ }n) < \alpha\}$ is $\mathcal{F}$-positive.

Claim: for every $s \in \omega^{<\omega}$, $\mathrm{rank}(s)$ defined (thus $< \omega_1$).

Suppose $\mathrm{rank}(s)$ undefined for some s.

Then $\{n : \mathrm{rank}(s\hat{\ }n)$ is undefined$\} \in \mathcal{F}$.

Recursively build tree $S \in \mathbb{L}_{\mathcal{F}}$ such that $\mathrm{stem}(S) = s$ and for all $t \supseteq s$ in S, $\mathrm{rank}(t)$ is undefined.

Absoluteness for Laver forcing 2

Forwards direction: rank argument!

Let $A \in M$ be a maximal antichain in $\mathbb{L}_{\mathcal{F}}$. By recursion on $\alpha < \omega_1$, define in M when $\mathrm{rank}(s) = \alpha$ for $s \in \omega^{<\omega}$.

- $\mathrm{rank}(s) = 0$ if $\exists T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
- $\mathrm{rank}(s) = \alpha$ if
  - there is no $\beta < \alpha$ with $\mathrm{rank}(s) = \beta$, and
  - $\{n : \mathrm{rank}(s\hat{\ }n) < \alpha\}$ is $\mathcal{F}$-positive.

Claim: for every $s \in \omega^{<\omega}$, $\mathrm{rank}(s)$ defined (thus $< \omega_1$).

Suppose $\mathrm{rank}(s)$ undefined for some s.

Then $\{n : \mathrm{rank}(s\hat{\ }n) \text{ is undefined}\} \in \mathcal{F}$.

Recursively build tree $S \in \mathbb{L}_{\mathcal{F}}$ such that $\mathrm{stem}(S) = s$ and for all $t \supseteq s$ in S, $\mathrm{rank}(t)$ is undefined.

Let $T \in A$ be compatible with S with common extension U.

Then: $\mathrm{stem}(T) \subseteq \mathrm{stem}(U) \in U \subseteq T$ so that $\mathrm{rank}(\mathrm{stem}(U)) = 0$.

Also: $\mathrm{stem}(S) \subseteq \mathrm{stem}(U) \in U \subseteq S$ so that $\mathrm{rank}(\mathrm{stem}(U))$ undef.

Absoluteness for Laver forcing 3

Let $S \in N$ be a condition in $\mathbb{L}_{\mathcal{G}}$. Put $s = \mathrm{stem}(S)$.

Absoluteness for Laver forcing 3

Let $S \in N$ be a condition in $\mathbb{L}_{\mathcal{G}}$. Put $s = \mathrm{stem}(S)$.
By induction on $\mathrm{rank}(s)$, show there is $T \in A$ compatible with S.

Absoluteness for Laver forcing 3

Let $S \in N$ be a condition in $\mathbb{L}_{\mathcal{G}}$. Put $s = \mathrm{stem}(S)$.
By induction on $\mathrm{rank}(s)$, show there is $T \in A$ compatible with S.

- $\mathrm{rank}(s) = 0$: there is $T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
  Compatibility: straightforward.

Absoluteness for Laver forcing 3

Let $S \in N$ be a condition in $\mathbb{L}_{\mathcal{G}}$. Put $s = \mathrm{stem}(S)$.
By induction on $\mathrm{rank}(s)$, show there is $T \in A$ compatible with S.

- $\mathrm{rank}(s) = 0$: there is $T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
  Compatibility: straightforward.
- $\mathrm{rank}(s) > 0$: Consider $\{n : \mathrm{rank}(s\hat{\ }n) < \mathrm{rank}(s)\}$.
  This set is $\mathcal{F}$-positive and, by assumption, still $\mathcal{G}$-positive.

Absoluteness for Laver forcing 3

Let $S \in N$ be a condition in $\mathbb{L}_{\mathcal{G}}$. Put $s = \mathrm{stem}(S)$.
By induction on $\mathrm{rank}(s)$, show there is $T \in A$ compatible with S.

- $\mathrm{rank}(s) = 0$: there is $T \in A$ such that $\mathrm{stem}(T) \subseteq s \in T$.
  Compatibility: straightforward.
- $\mathrm{rank}(s) > 0$: Consider $\{n : \mathrm{rank}(s\hat{\ }n) < \mathrm{rank}(s)\}$.
  This set is $\mathcal{F}$-positive and, by assumption, still $\mathcal{G}$-positive.
  Hence there is $n \in \mathrm{succ}_S(s)$ with $\mathrm{rank}(s\hat{\ }n) < \mathrm{rank}(s)$.
  Consider $S_{s\hat{\ }n} = \{t \in S : t \subseteq s \text{ or } s\hat{\ }n \subseteq t\}$.
  This is a subtree of S with stem $s\hat{\ }n$.
  By induction hypothesis, there is $T \in A$ compatible with $S_{s\hat{\ }n}$.
  But then T is also compatible with S. $\square$

Absoluteness for Laver and Mathias forcing

Corollary (Shelah)

Let $\mathcal{U}$ be an ultrafilter in M and let $\mathcal{V}$ be an ultrafilter in N extending $\mathcal{U}$. Then every maximal antichain of $\mathbb{L}_{\mathcal{U}}$ in M is still a maximal antichain of $\mathbb{L}_{\mathcal{V}}$ in N.

Absoluteness for Laver and Mathias forcing

Corollary (Shelah)

Let $\mathcal{U}$ be an ultrafilter in M and let $\mathcal{V}$ be an ultrafilter in N extending $\mathcal{U}$. Then every maximal antichain of $\mathbb{L}_\mathcal{U}$ in M is still a maximal antichain of $\mathbb{L}_\mathcal{V}$ in N.

Even this special case fails for Mathias forcing:

Example

Assume $\mathcal{U} \in M$ is not Ramsey, and assume there is a Cohen real in N over M. Then there are an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N and a maximal antichain $A \subseteq \mathbb{M}_\mathcal{U}$ in M which is not maximal in $\mathbb{M}_\mathcal{V}$.

Absoluteness for Mathias forcing

On the other hand, given an arbitrary $\mathcal{U}$, we can always find $\mathcal{V}$ such that maximal antichains are preserved:

Absoluteness for Mathias forcing

On the other hand, given an arbitrary $\mathcal{U}$, we can always find $\mathcal{V}$ such that maximal antichains are preserved:

Lemma (Blass-Shelah)

Let $\mathcal{U}$ be an ultrafilter in M.
Also assume there is $c \in \omega^\omega \cap N$ unbounded over M.
Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that:

(i) *every maximal antichain of $\mathbb{M}_\mathcal{U}$ in M is still a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N*

(ii) *c is unbounded over $M^{\mathbb{M}_\mathcal{U}}$ in $N^{\mathbb{M}_\mathcal{V}}$.*

Complete embeddability

Using these absoluteness results

- we obtain complete embeddability
- we build long iterations which can be realized as direct limits of "short iterations"

as in lecture 1.

Complete embeddability

Using these absoluteness results

- we obtain complete embeddability
- we build long iterations which can be realized as direct limits of "short iterations"

as in lecture 1. Recall from lecture 1:

Lemma (preservation of embeddability in iterations)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s. Let $\dot{\mathbb{Q}}_i$ be $\mathbb{P}_i$-names for p.o.'s such that $\mathbb{P}_1 \Vdash \dot{\mathbb{Q}}_0 \subseteq \dot{\mathbb{Q}}_1$ and all maximal antichains of $\dot{\mathbb{Q}}_0$ in $V^{\mathbb{P}_0}$ are maximal antichains of $\dot{\mathbb{Q}}_1$ in $V^{\mathbb{P}_1}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0 \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$.

In our context, this means:

Complete embeddability

Using these absoluteness results

- we obtain complete embeddability
- we build long iterations which can be realized as direct limits of "short iterations"

as in lecture 1.

Lemma (preservation of embeddability in iterations)

Let $\mathbb{P}_0 \lessdot \mathbb{P}_1$ be p.o.'s. Let $\dot{\mathcal{F}}_i$ be $\mathbb{P}_i$-names for filters such that $\mathbb{P}_1 \Vdash \dot{\mathcal{F}}_0 \subseteq \dot{\mathcal{F}}_1$ and all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_0}$ in $V^{\mathbb{P}_0}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_1}$ in $V^{\mathbb{P}_1}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$.
Then $\mathbb{P}_0 \star \dot{\mathbb{X}}_{\dot{\mathcal{F}}_0} \lessdot \mathbb{P}_1 \star \dot{\mathbb{X}}_{\dot{\mathcal{F}}_1}$.

Matrices: the first step 1

Let $\mu < \lambda$ be uncountable regular cardinals.
Assume $(\mathbb{P}_0^\gamma : \gamma \leq \mu)$ is a ccc iteration such that $\mathbb{P}_0^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_0^\gamma$.

Matrices: the first step 1

Let $\mu < \lambda$ be uncountable regular cardinals.
Assume $(\mathbb{P}_0^\gamma : \gamma \leq \mu)$ is a ccc iteration such that $\mathbb{P}_0^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_0^\gamma$.

By recursion on γ choose $\mathbb{P}_0^\gamma$-names for filters $\dot{\mathcal{F}}_0^\gamma$ such that

- $\mathbb{P}_0^\delta \Vdash \dot{\mathcal{F}}_0^\gamma \subseteq \dot{\mathcal{F}}_0^\delta$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$ in $V^{\mathbb{P}_0^\gamma}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$ in $V^{\mathbb{P}_0^\delta}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

Matrices: the first step 1

Let $\mu < \lambda$ be uncountable regular cardinals.
Assume $(\mathbb{P}_0^\gamma : \gamma \leq \mu)$ is a ccc iteration such that $\mathbb{P}_0^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_0^\gamma$.

By recursion on γ choose $\mathbb{P}_0^\gamma$-names for filters $\dot{\mathcal{F}}_0^\gamma$ such that

- $\mathbb{P}_0^\delta \Vdash \dot{\mathcal{F}}_0^\gamma \subseteq \dot{\mathcal{F}}_0^\delta$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$ in $V^{\mathbb{P}_0^\gamma}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$ in $V^{\mathbb{P}_0^\delta}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

Then let $\mathbb{P}_1^\gamma = \mathbb{P}_0^\gamma \star \mathbb{X}_{\dot{\mathcal{F}}_1^\gamma}$.

Matrices: the first step 2

Properties:

- if x is $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$-generic over $V^{\mathbb{P}_0^\delta}$, then it is also $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$-generic over $V^{\mathbb{P}_0^\gamma}$ for $\gamma < \delta$
(by preservation of maximal antichains)

Matrices: the first step 2

Properties:

- if x is $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$-generic over $V^{\mathbb{P}_0^\delta}$, then it is also $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$-generic over $V^{\mathbb{P}_0^\gamma}$ for $\gamma < \delta$
  (by preservation of maximal antichains)
- $\mathbb{P}_1^\gamma \lessdot \mathbb{P}_1^\delta$ for $\gamma < \delta$
  (by preservation of embeddability)

Matrices: the first step 2

Properties:

- if x is $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$-generic over $V^{\mathbb{P}_0^\delta}$, then it is also $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$-generic over $V^{\mathbb{P}_0^\gamma}$ for $\gamma < \delta$
  (by preservation of maximal antichains)
- $\mathbb{P}_1^\gamma \lessdot \mathbb{P}_1^\delta$ for $\gamma < \delta$
  (by preservation of embeddability)
- $\mathbb{P}_1^\mu = \lim\mathrm{dir}_{\gamma<\mu}\mathbb{P}_1^\gamma$

Matrices: the first step 2

Properties:

- if x is $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$-generic over $V^{\mathbb{P}_0^\delta}$, then it is also $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$-generic over $V^{\mathbb{P}_0^\gamma}$ for $\gamma < \delta$
  (by preservation of maximal antichains)
- $\mathbb{P}_1^\gamma \lessdot \mathbb{P}_1^\delta$ for $\gamma < \delta$
  (by preservation of embeddability)
- $\mathbb{P}_1^\mu = \lim\mathrm{dir}_{\gamma<\mu}\mathbb{P}_1^\gamma$
- $V_1^\mu \cap \omega^\omega = \bigcup_{\gamma<\mu} V_1^\gamma \cap \omega^\omega$

Matrices: the first step 2

Properties:

- if x is $\mathbb{X}_{\dot{\mathcal{F}}_0^\delta}$-generic over $V^{\mathbb{P}_0^\delta}$, then it is also $\mathbb{X}_{\dot{\mathcal{F}}_0^\gamma}$-generic over $V^{\mathbb{P}_0^\gamma}$ for $\gamma < \delta$
(by preservation of maximal antichains)
- $\mathbb{P}_1^\gamma <\!\circ\, \mathbb{P}_1^\delta$ for $\gamma < \delta$
(by preservation of embeddability)
- $\mathbb{P}_1^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_1^\gamma$
- $V_1^\mu \cap \omega^\omega = \bigcup_{\gamma<\mu} V_1^\gamma \cap \omega^\omega$

In particular, $(\mathbb{P}_1^\gamma : \gamma \leq \mu)$ is a ccc iteration such that $\mathbb{P}_1^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_1^\gamma$.

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}_\alpha^\gamma : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}_\alpha^\gamma <\!\circ\ \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}_\alpha^\gamma : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(ii) $\mathbb{P}_\alpha^\mu = \lim\operatorname{dir}_{\gamma<\mu} \mathbb{P}_\alpha^\gamma$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}_\alpha^\gamma : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(ii) $\mathbb{P}_\alpha^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_\alpha^\gamma$

(iii) $V_\alpha^\mu \cap \omega^\omega = \bigcup_{\gamma<\mu} V_\alpha^\gamma \cap \omega^\omega$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}^\gamma_\alpha : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(ii) $\mathbb{P}^\mu_\alpha = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}^\gamma_\alpha$

(iii) $V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}^\gamma_\alpha : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(ii) $\mathbb{P}^\mu_\alpha = \lim\mathrm{dir}_{\gamma<\mu} \mathbb{P}^\gamma_\alpha$

(iii) $V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that

- $\mathbb{P}^\delta_\alpha \Vdash \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}^\gamma_\alpha : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(ii) $\mathbb{P}^\mu_\alpha = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}^\gamma_\alpha$

(iii) $V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that

- $\mathbb{P}^\delta_\alpha \Vdash \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\gamma_\alpha}$ in $V^{\mathbb{P}^\gamma_\alpha}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\delta_\alpha}$ in $V^{\mathbb{P}^\delta_\alpha}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}_\alpha^\gamma : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(ii) $\mathbb{P}_\alpha^\mu = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}_\alpha^\gamma$

(iii) $V_\alpha^\mu \cap \omega^\omega = \bigcup_{\gamma<\mu} V_\alpha^\gamma \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}_\alpha^\gamma$-names for filters $\dot{\mathcal{F}}_\alpha^\gamma$ such that

- $\mathbb{P}_\alpha^\delta \Vdash \dot{\mathcal{F}}_\alpha^\gamma \subseteq \dot{\mathcal{F}}_\alpha^\delta$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_\alpha^\gamma}$ in $V^{\mathbb{P}_\alpha^\gamma}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}_\alpha^\delta}$ in $V^{\mathbb{P}_\alpha^\delta}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

and we put $\mathbb{P}_\beta^\gamma = \mathbb{P}_\alpha^\gamma \star \mathbb{X}_{\dot{\mathcal{F}}_\alpha^\gamma}$

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}^\gamma_\alpha : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(ii) $\mathbb{P}^\mu_\alpha = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}^\gamma_\alpha$

(iii) $V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that

- $\mathbb{P}^\delta_\alpha \Vdash \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\gamma_\alpha}$ in $V^{\mathbb{P}^\gamma_\alpha}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\delta_\alpha}$ in $V^{\mathbb{P}^\delta_\alpha}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

and we put $\mathbb{P}^\gamma_\beta = \mathbb{P}^\gamma_\alpha \star \mathbb{X}_{\dot{\mathcal{F}}^\gamma_\alpha}$

Successor step $\beta = \alpha + 1$: like $\beta = 1$ of previous slide.

Matrices: the general case

More generally, by recursion on $\alpha < \lambda$, build finite support iterations $(\mathbb{P}^\gamma_\alpha : \alpha \leq \lambda)$, $\gamma \leq \mu$, such that

(i) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(ii) $\mathbb{P}^\mu_\alpha = \lim \mathrm{dir}_{\gamma<\mu} \mathbb{P}^\gamma_\alpha$

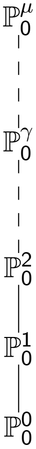
(iii) $V^\mu_\alpha \cap \omega^\omega = \bigcup_{\gamma<\mu} V^\gamma_\alpha \cap \omega^\omega$

(iv) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that
   - $\mathbb{P}^\delta_\alpha \Vdash \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$
   - all maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\gamma_\alpha}$ in $V^{\mathbb{P}^\gamma_\alpha}$ are maximal antichains of $\mathbb{X}_{\dot{\mathcal{F}}^\delta_\alpha}$ in $V^{\mathbb{P}^\delta_\alpha}$ where $\mathbb{X} = \mathbb{L}, \mathbb{M}$

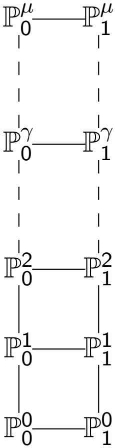
   and we put $\mathbb{P}^\gamma_\beta = \mathbb{P}^\gamma_\alpha \star \mathbb{X}_{\dot{\mathcal{F}}^\gamma_\alpha}$

Successor step $\beta = \alpha + 1$: like $\beta = 1$ of previous slide.
Limit step: (i), (ii), (iii): exercise!

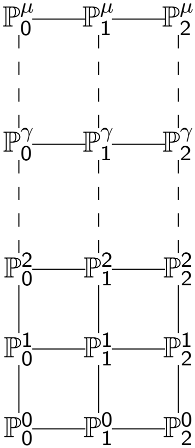
Matrices: a diagram



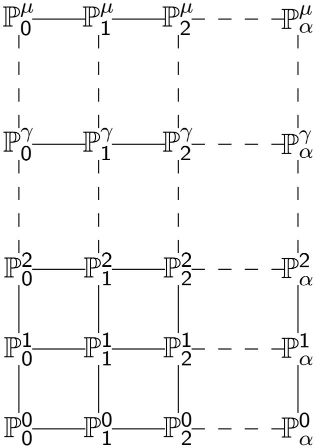
Matrices: a diagram



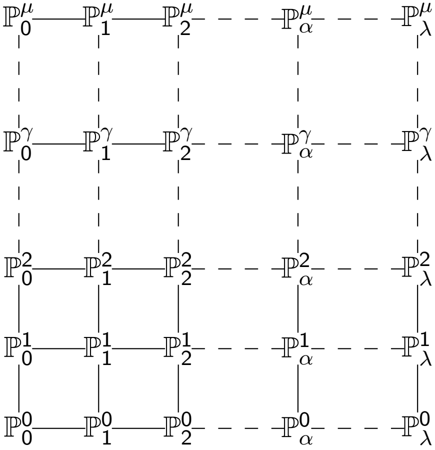
Matrices: a diagram



Matrices: a diagram



Matrices: a diagram



Dense sets of rationals

Let $\mathbf{Q}$ denote the rationals.
$\mathrm{Dense}(\mathbf{Q})$: dense subsets of rationals.
nwd: nowhere dense sets of rationals

Dense sets of rationals

Let $\mathbf{Q}$ denote the rationals.
$\mathrm{Dense}(\mathbf{Q})$: dense subsets of rationals.
nwd: nowhere dense sets of rationals

For $A, B \in \mathrm{Dense}(\mathbf{Q})$:

$$A \subseteq_{\mathrm{nwd}} B \quad (A \text{ is contained in } B \bmod \mathrm{nwd}) \iff A \setminus B \in \mathrm{nwd}$$

Dense sets of rationals

Let $\mathbf{Q}$ denote the rationals.
$\mathrm{Dense}(\mathbf{Q})$: dense subsets of rationals.
nwd: nowhere dense sets of rationals

For $A, B \in \mathrm{Dense}(\mathbf{Q})$:

$$A \subseteq_{\mathrm{nwd}} B \quad (A \text{ is contained in } B \text{ mod } \mathrm{nwd}) \iff A \setminus B \in \mathrm{nwd}$$

Consider the quotient $\mathrm{Dense}(\mathbf{Q})/\mathrm{nwd}$ ordered by
$[A] \leq [B]$ iff $A \subseteq_{\mathrm{nwd}} B$.

Cardinal invariants for $\mathrm{Dense}(\mathbf{Q})/\mathrm{nwd}$ 1

For $A, B \in \mathrm{Dense}(\mathbf{Q})$:

A *$\mathbf{Q}$-splits* $B \iff A \cap B$ and $B \setminus A$ both dense

Cardinal invariants for $\mathrm{Dense}(\mathbf{Q})/\mathrm{nwd}$ 1

For $A, B \in \mathrm{Dense}(\mathbf{Q})$:

$$A \ \mathbf{Q}\textit{-splits}\ B \iff A \cap B \text{ and } B \setminus A \text{ both dense}$$

$\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-*splitting* if every member of $\mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-*unsplit* (or $\mathbf{Q}$-*unreaped*) if no member of $\mathrm{Dense}(\mathbf{Q})$ $\mathbf{Q}$-splits all members of $\mathcal{F}$, i.e.
$\forall A \in \mathrm{Dense}(\mathbf{Q})\ \exists B \in \mathcal{F}\ (A \cap B \text{ not dense or } B \setminus A \text{ not dense})$.

Cardinal invariants for $\mathrm{Dense}(\mathbf{Q})/\mathrm{nwd}$ 1

For $A, B \in \mathrm{Dense}(\mathbf{Q})$:

$$A \ \mathbf{Q}\text{-}\mathit{splits}\ B \iff A \cap B \text{ and } B \setminus A \text{ both dense}$$

$\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-*splitting* if every member of $\mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-split by a member of $\mathcal{F}$.
$\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is $\mathbf{Q}$-*unsplit* (or $\mathbf{Q}$-*unreaped*) if no member of $\mathrm{Dense}(\mathbf{Q})$ $\mathbf{Q}$-splits all members of $\mathcal{F}$, i.e.
$\forall A \in \mathrm{Dense}(\mathbf{Q})\ \exists B \in \mathcal{F}\ (A \cap B \text{ not dense or } B \setminus A \text{ not dense})$.

$\mathfrak{s}_{\mathbf{Q}} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathbf{Q}\text{-splitting}\}$, the $\mathbf{Q}$-*splitting number*.
$\mathfrak{r}_{\mathbf{Q}} := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathbf{Q}\text{-unsplit}\}$, the $\mathbf{Q}$-*reaping number*.

Cardinal invariants for $\mathrm{Dense}(\mathbb{Q})/\mathrm{nwd}$ 2

$\mathcal{D} \subseteq \mathrm{Dense}(\mathbb{Q})$ $\mathbb{Q}$-*dense*: $\forall A \in \mathrm{Dense}(\mathbb{Q}) \;\; \exists B \in \mathcal{D} \;\; (B \subseteq_{\mathrm{nwd}} A)$

Cardinal invariants for $\mathrm{Dense}(\mathbf{Q})/\mathrm{nwd}$ 2

$\mathcal{D} \subseteq \mathrm{Dense}(\mathbf{Q})$ $\mathbf{Q}$-*dense*: $\forall A \in \mathrm{Dense}(\mathbf{Q}) \; \exists B \in \mathcal{D} \; (B \subseteq_{\mathrm{nwd}} A)$

$\mathfrak{h}_{\mathbf{Q}} := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ open } \mathbf{Q}\text{-dense and } \bigcap \mathfrak{D} = \emptyset\}$
the $\mathbf{Q}$-*distributivity number*.

Cardinal invariants for $\text{Dense}(\mathbf{Q})/\text{nwd}$ 2

$\mathcal{D} \subseteq \text{Dense}(\mathbf{Q})$ $\mathbf{Q}$-*dense*: $\forall A \in \text{Dense}(\mathbf{Q})\ \exists B \in \mathcal{D}\ (B \subseteq_{\text{nwd}} A)$

$\mathfrak{h}_{\mathbf{Q}} := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ open } \mathbf{Q}\text{-dense and } \bigcap \mathfrak{D} = \emptyset\}$
the $\mathbf{Q}$-*distributivity number*.

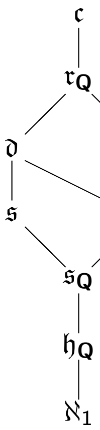
Let $\mathcal{M}$ be the meager ideal.

Theorem

(i) $\mathfrak{s}_{\mathbf{Q}} \leq \min\{\mathfrak{s}, \text{add}(\mathcal{M})\} \leq \min\{\mathfrak{s}, \mathfrak{b}\}$ *and dually*
$\max\{\mathfrak{r}, \mathfrak{d}\} \leq \max\{\mathfrak{r}, \text{cof}(\mathcal{M})\} \leq \mathfrak{r}_{\mathbf{Q}}$

(ii) $\mathfrak{h}_{\mathbf{Q}} \leq \mathfrak{s}_{\mathbf{Q}}$

ZFC-inequalities: another diagram



First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 1

Theorem (B.)

Let $\lambda = \lambda^\omega$ be regular uncountable. It is consistent that $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$ and $\mathfrak{h}_{\mathbf{Q}} = \aleph_1$.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 1

Theorem (B.)

Let $\lambda = \lambda^\omega$ be regular uncountable. It is consistent that $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$ and $\mathfrak{h}_{\mathbf{Q}} = \aleph_1$.

Proof: $\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is a *maximal* $\mathbf{Q}$*-filter* if $\mathcal{F}$ is a filter in $\mathrm{Dense}(\mathbf{Q})$ which cannot be extended to a strictly larger filter in $\mathrm{Dense}(\mathbf{Q})$.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 1

Theorem (B.)

Let $\lambda = \lambda^\omega$ be regular uncountable. It is consistent that $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$ and $\mathfrak{h}_{\mathbf{Q}} = \aleph_1$.

Proof: $\mathcal{F} \subseteq \mathrm{Dense}(\mathbf{Q})$ is a *maximal* **Q***-filter* if $\mathcal{F}$ is a filter in $\mathrm{Dense}(\mathbf{Q})$ which cannot be extended to a strictly larger filter in $\mathrm{Dense}(\mathbf{Q})$.

Fact: If $N \subseteq M$, $\mathcal{F}$ is a maximal **Q**-filter in M and $\mathcal{G}$ is a maximal **Q**-filter in N extending $\mathcal{F}$, then every $\mathcal{F}$-positive set of M is $\mathcal{G}$-positive in N.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing.

Lemma (preservation of maximal antichains)

The following are equivalent:

(i) *every $\mathcal{F}$-positive set in M is still $\mathcal{G}$-positive in N*

(ii) *every maximal antichain of $\mathbb{L}_{\mathcal{F}}$ in M is still a maximal antichain of $\mathbb{L}_{\mathcal{G}}$ in N*

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(iii) $\mathbb{P}_\alpha^{\aleph_1} = \lim \mathrm{dir}_{\gamma < \aleph_1} \mathbb{P}_\alpha^\gamma$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(iii) $\mathbb{P}_\alpha^{\aleph_1} = \lim\mathrm{dir}_{\gamma<\aleph_1} \mathbb{P}_\alpha^\gamma$

(iv) $V_\alpha^{\aleph_1} \cap \omega^\omega = \bigcup_{\gamma<\aleph_1} (V_\alpha^\gamma \cap \omega^\omega)$ and $\omega^\omega \cap (V_\alpha^\delta \setminus V_\alpha^\gamma) \neq \emptyset$ for $\gamma < \delta$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(iii) $\mathbb{P}_\alpha^{\aleph_1} = \text{lim dir}_{\gamma<\aleph_1} \mathbb{P}_\alpha^\gamma$

(iv) $V_\alpha^{\aleph_1} \cap \omega^\omega = \bigcup_{\gamma<\aleph_1} (V_\alpha^\gamma \cap \omega^\omega)$ and $\omega^\omega \cap (V_\alpha^\delta \setminus V_\alpha^\gamma) \neq \emptyset$ for $\gamma < \delta$

(v) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}_\alpha^\gamma$-names for maximal **Q**-filters $\dot{\mathcal{F}}_\alpha^\gamma$ such that

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}^\gamma_\alpha$ being maximal **Q**-filters:

(i) $\mathbb{P}^\gamma_0 = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(iii) $\mathbb{P}^{\aleph_1}_\alpha = \lim\mathrm{dir}_{\gamma<\aleph_1} \mathbb{P}^\gamma_\alpha$

(iv) $V^{\aleph_1}_\alpha \cap \omega^\omega = \bigcup_{\gamma<\aleph_1} (V^\gamma_\alpha \cap \omega^\omega)$ and $\omega^\omega \cap (V^\delta_\alpha \setminus V^\gamma_\alpha) \neq \emptyset$ for $\gamma < \delta$

(v) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for maximal **Q**-filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that

- $\Vdash^\delta_\alpha \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}_\alpha^\gamma$ being maximal **Q**-filters:

(i) $\mathbb{P}_0^\gamma = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}_\alpha^\gamma \lessdot \mathbb{P}_\alpha^\delta$ for $\gamma < \delta$

(iii) $\mathbb{P}_\alpha^{\aleph_1} = \text{lim dir}_{\gamma < \aleph_1} \mathbb{P}_\alpha^\gamma$

(iv) $V_\alpha^{\aleph_1} \cap \omega^\omega = \bigcup_{\gamma < \aleph_1} (V_\alpha^\gamma \cap \omega^\omega)$ and $\omega^\omega \cap (V_\alpha^\delta \setminus V_\alpha^\gamma) \neq \emptyset$ for $\gamma < \delta$

(v) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}_\alpha^\gamma$-names for maximal **Q**-filters $\dot{\mathcal{F}}_\alpha^\gamma$ such that
- $\Vdash_\alpha^\delta \dot{\mathcal{F}}_\alpha^\gamma \subseteq \dot{\mathcal{F}}_\alpha^\delta$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{L}_{\dot{\mathcal{F}}_\alpha^\gamma}$ in $V^{\mathbb{P}_\alpha^\gamma}$ are maximal antichains of $\mathbb{L}_{\dot{\mathcal{F}}_\alpha^\delta}$ in $V^{\mathbb{P}_\alpha^\delta}$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 2

So we may apply preservation of maximal antichains for Laver forcing. This allows us to build a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{L}$ and the $\dot{\mathcal{F}}^\gamma_\alpha$ being maximal **Q**-filters:

(i) $\mathbb{P}^\gamma_0 = \mathbb{C}^\gamma$ adds γ Cohen reals

(ii) $\mathbb{P}^\gamma_\alpha \lessdot \mathbb{P}^\delta_\alpha$ for $\gamma < \delta$

(iii) $\mathbb{P}^{\aleph_1}_\alpha = \lim \mathrm{dir}_{\gamma < \aleph_1} \mathbb{P}^\gamma_\alpha$

(iv) $V^{\aleph_1}_\alpha \cap \omega^\omega = \bigcup_{\gamma < \aleph_1} (V^\gamma_\alpha \cap \omega^\omega)$ and $\omega^\omega \cap (V^\delta_\alpha \setminus V^\gamma_\alpha) \neq \emptyset$ for $\gamma < \delta$

(v) if $\beta = \alpha + 1$ is a successor, we have $\mathbb{P}^\gamma_\alpha$-names for maximal **Q**-filters $\dot{\mathcal{F}}^\gamma_\alpha$ such that
- $\Vdash^\delta_\alpha \dot{\mathcal{F}}^\gamma_\alpha \subseteq \dot{\mathcal{F}}^\delta_\alpha$ for $\gamma < \delta$
- all maximal antichains of $\mathbb{L}_{\dot{\mathcal{F}}^\gamma_\alpha}$ in $V^{\mathbb{P}^\gamma_\alpha}$ are maximal antichains of $\mathbb{L}_{\dot{\mathcal{F}}^\delta_\alpha}$ in $V^{\mathbb{P}^\delta_\alpha}$

and we put $\mathbb{P}^\gamma_\beta = \mathbb{P}^\gamma_\alpha \star \mathbb{L}_{\dot{\mathcal{F}}^\gamma_\alpha}$

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 3

Fact: Let $\mathcal{F}$ be a maximal $\mathbf{Q}$-filter.
If ℓ is $\mathbb{L}_{\mathcal{F}}$-generic over V, $\mathrm{ran}(\ell)$ is not $\mathbf{Q}$-split by any ground model dense set.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 3

Fact: Let $\mathcal{F}$ be a maximal **Q**-filter.
If ℓ is $\mathbb{L}_{\mathcal{F}}$-generic over V, $\mathrm{ran}(\ell)$ is not **Q**-split by any ground model dense set.

Since we iterate λ times, $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 3

Fact: Let $\mathcal{F}$ be a maximal **Q**-filter.
If ℓ is $\mathbb{L}_{\mathcal{F}}$-generic over V, $\mathrm{ran}(\ell)$ is not **Q**-split by any ground model dense set.

Since we iterate λ times, $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$.

Lemma

Let κ be an uncountable cardinal. Assume there is an increasing chain of ZFC-models V_α, $\alpha < \kappa$, such that

(i) $\omega^\omega \cap V = \bigcup_{\alpha<\kappa}(\omega^\omega \cap V_\alpha)$

(ii) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$ *for all* $\alpha < \kappa$.

Then $\mathfrak{h}_{\mathbf{Q}} \leq \kappa$.

First application: $\mathfrak{h}_{\mathbf{Q}}$ versus $\mathfrak{s}_{\mathbf{Q}}$ 3

Fact: Let $\mathcal{F}$ be a maximal $\mathbf{Q}$-filter.
If ℓ is $\mathbb{L}_{\mathcal{F}}$-generic over V, $\mathrm{ran}(\ell)$ is not $\mathbf{Q}$-split by any ground model dense set.

Since we iterate λ times, $\mathfrak{s}_{\mathbf{Q}} = \mathfrak{c} = \lambda$.

Lemma

Let κ be an uncountable cardinal. Assume there is an increasing chain of ZFC-models V_α, $\alpha < \kappa$, such that

(i) $\omega^\omega \cap V = \bigcup_{\alpha<\kappa}(\omega^\omega \cap V_\alpha)$

(ii) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$ *for all* $\alpha < \kappa$.

Then $\mathfrak{h}_{\mathbf{Q}} \leq \kappa$.

By (iv): true with $\kappa = \aleph_1$, $V = V_\lambda^{\aleph_1}$ and $V_\alpha = V_\lambda^\alpha$.
Hence: $\mathfrak{h}_{\mathbf{Q}} = \aleph_1$. $\square$

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Blass-Shelah)

Let $\lambda = \lambda^\omega$ be regular uncountable. It is consistent that $\mathfrak{s} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \aleph_1$.

Second application: $\mathfrak{b}$ versus $\mathfrak{s}$

Theorem (Blass-Shelah)

Let $\lambda = \lambda^\omega$ be regular uncountable. It is consistent that $\mathfrak{s} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \aleph_1$.

Use a matrix iteration with $\mu = \aleph_1$, $\mathbb{X} = \mathbb{M}$ and the $\dot{\mathcal{U}}_\alpha^\gamma$ being ultrafilters. Recall:

Lemma (Blass-Shelah)

Let $\mathcal{U}$ be an ultrafilter in M.
Also assume there is $c \in \omega^\omega \cap N$ unbounded over M.
Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that:

(i) *every maximal antichain of $\mathbb{M}_\mathcal{U}$ in M is still a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N*

(ii) *c is unbounded over $M^{\mathbb{M}_\mathcal{U}}$ in $N^{\mathbb{M}_\mathcal{V}}$.*

Ultrapowers of p.o.'s

κ: measurable cardinal
$\mathcal{D}$: κ-complete ultrafilter on κ

Ultrapowers of p.o.'s

κ: measurable cardinal

$\mathcal{D}$: κ-complete ultrafilter on κ

Let $\mathbb{P}$ be a p.o. and consider the *ultrapower*

$$\mathbb{P}^\kappa/\mathcal{D} = \{[f] : f : \kappa \to \mathbb{P}\}$$

where $[f] = \{g \in \mathbb{P}^\kappa : \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$ is the equivalence class of f.

Ultrapowers of p.o.'s

κ: measurable cardinal
$\mathcal{D}$: κ-complete ultrafilter on κ
Let $\mathbb{P}$ be a p.o. and consider the *ultrapower*

$$\mathbb{P}^\kappa/\mathcal{D} = \{[f] : f : \kappa \to \mathbb{P}\}$$

where $[f] = \{g \in \mathbb{P}^\kappa : \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$ is the equivalence class of f.
$\mathbb{P}^\kappa/\mathcal{D}$ is ordered by

$$[g] \leq [f] \text{ if } \{\alpha < \kappa : g(\alpha) \leq f(\alpha)\} \in \mathcal{D}$$

Identifying $p \in \mathbb{P}$ with the class $[f]$ of the constant function $f(\alpha) = p$ for all α, we may assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathcal{D}$.

Complete embeddability

Lemma (Complete embeddability)

Let $A \subseteq \mathbb{P}$ be a maximal antichain.
Then A is maximal in $\mathbb{P}^\kappa/\mathcal{D}$ iff $|A| < \kappa$.
In particular, $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{D}$ iff $\mathbb{P}$ has the κ-cc.

Complete embeddability

Lemma (Complete embeddability)

Let $A \subseteq \mathbb{P}$ be a maximal antichain.
Then A is maximal in $\mathbb{P}^\kappa / \mathcal{D}$ iff $|A| < \kappa$.
In particular, $\mathbb{P} <\circ \mathbb{P}^\kappa / \mathcal{D}$ iff $\mathbb{P}$ has the κ-cc.

Proof: A: an antichain of $\mathbb{P}$ of size at least κ.
f: any injection from κ into A.
Then: $[f]$ is incompatible with all members of A.

Complete embeddability

Lemma (Complete embeddability)

Let $A \subseteq \mathbb{P}$ be a maximal antichain.
Then A is maximal in $\mathbb{P}^\kappa/\mathcal{D}$ iff $|A| < \kappa$.
In particular, $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{D}$ iff $\mathbb{P}$ has the κ-cc.

Proof: A: an antichain of $\mathbb{P}$ of size at least κ.
f: any injection from κ into A.
Then: $[f]$ is incompatible with all members of A.

Let A be an antichain of $\mathbb{P}$ of size $< \kappa$.
Assume $[f] \in \mathbb{P}^\kappa/\mathcal{D}$ is incompatible with all members of A.
For $p \in A$: $X_p := \{\alpha : f(\alpha) \text{ and } p \text{ are incompatible}\} \in \mathcal{D}$.

Complete embeddability

Lemma (Complete embeddability)

Let $A \subseteq \mathbb{P}$ be a maximal antichain.
Then A is maximal in $\mathbb{P}^\kappa/\mathcal{D}$ iff $|A| < \kappa$.
In particular, $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{D}$ iff $\mathbb{P}$ has the κ-cc.

Proof: A: an antichain of $\mathbb{P}$ of size at least κ.
f: any injection from κ into A.
Then: $[f]$ is incompatible with all members of A.

Let A be an antichain of $\mathbb{P}$ of size $< \kappa$.
Assume $[f] \in \mathbb{P}^\kappa/\mathcal{D}$ is incompatible with all members of A.
For $p \in A$: $X_p := \{\alpha : f(\alpha) \text{ and } p \text{ are incompatible}\} \in \mathcal{D}$.
κ-completeness: $X := \bigcap_{p\in A} X_p \in \mathcal{D}$.
If $\alpha \in X$: $f(\alpha)$ is incompatible with all $p \in A$. $\square$

Preservation of chain condition

Lemma (Preservation of the chain condition)

Assume $\mathbb{P}$ has the λ-cc for some $\lambda < \kappa$.
Then $\mathbb{P}^\kappa/\mathcal{D}$ has the λ-cc as well.

Preservation of chain condition

Lemma (Preservation of the chain condition)

*Assume $\mathbb{P}$ has the λ-cc for some $\lambda < \kappa$.
Then $\mathbb{P}^\kappa/\mathcal{D}$ has the λ-cc as well.*

Proof: Assume $[f_\gamma]$, $\gamma < \lambda$, pairwise incompatible in $\mathbb{P}^\kappa/\mathcal{D}$.
For $\gamma, \delta < \lambda$: $Y_{\gamma,\delta} := \{\alpha : f_\gamma(\alpha)$ and $f_\delta(\alpha)$ are incompatible$\} \in \mathcal{D}$.

Preservation of chain condition

Lemma (Preservation of the chain condition)

Assume $\mathbb{P}$ has the λ-cc for some $\lambda < \kappa$.
Then $\mathbb{P}^\kappa/\mathcal{D}$ has the λ-cc as well.

Proof: Assume $[f_\gamma]$, $\gamma < \lambda$, pairwise incompatible in $\mathbb{P}^\kappa/\mathcal{D}$.
For $\gamma, \delta < \lambda$: $Y_{\gamma,\delta} := \{\alpha : f_\gamma(\alpha)$ and $f_\delta(\alpha)$ are incompatible$\} \in \mathcal{D}$.
κ-completeness: $Y := \bigcap_{\gamma,\delta} Y_{\gamma,\delta} \in \mathcal{D}$.
If $\alpha \in Y$: $f_\gamma(\alpha)$, $\gamma < \lambda$, is an antichain in $\mathbb{P}$.
Contradiction to the λ-cc. $\square$

Preservation of chain condition

Lemma (Preservation of the chain condition)

Assume $\mathbb{P}$ has the λ-cc for some $\lambda < \kappa$.
Then $\mathbb{P}^\kappa/\mathcal{D}$ has the λ-cc as well.

Proof: Assume $[f_\gamma]$, $\gamma < \lambda$, pairwise incompatible in $\mathbb{P}^\kappa/\mathcal{D}$.
For $\gamma, \delta < \lambda$: $Y_{\gamma,\delta} := \{\alpha : f_\gamma(\alpha)$ and $f_\delta(\alpha)$ are incompatible$\} \in \mathcal{D}$.
κ-completeness: $Y := \bigcap_{\gamma,\delta} Y_{\gamma,\delta} \in \mathcal{D}$.
If $\alpha \in Y$: $f_\gamma(\alpha)$, $\gamma < \lambda$, is an antichain in $\mathbb{P}$.
Contradiction to the λ-cc. $\square$

Remark: If $\mathbb{P}$ has the κ-cc but not the λ-cc for any $\lambda < \kappa$, then $\mathbb{P}^\kappa/\mathcal{D}$ does not have the κ-cc.

Antichains and names for reals 1

Assume $\mathbb{P}$ is ccc.
Since $\mathbb{P}$ completely embeds into $\mathbb{P}^\kappa/\mathcal{D}$, we may write

$$\mathbb{P}^\kappa/\mathcal{D} = \mathbb{P} \star \dot{\mathbb{Q}}.$$

What can we say about the remainder forcing $\dot{\mathbb{Q}}$?
E.g., what kind of reals can it add?

Antichains and names for reals 1

Assume $\mathbb{P}$ is ccc.
Since $\mathbb{P}$ completely embeds into $\mathbb{P}^\kappa/\mathcal{D}$, we may write

$$\mathbb{P}^\kappa/\mathcal{D} = \mathbb{P} \star \dot{\mathbb{Q}}.$$

What can we say about the remainder forcing $\dot{\mathbb{Q}}$?
E.g., what kind of reals can it add?

Assume $\{[f_n] : n \in \omega\}$ is a maximal antichain in $\mathbb{P}^\kappa/\mathcal{D}$.
Know: $\{\alpha : \{f_n(\alpha) : n \in \omega\}$ is a maximal antichain$\} \in \mathcal{D}$.
Thus, by changing the f_n on a small set, we may as well assume that for all α, the $f_n(\alpha)$ form a maximal antichain in $\mathbb{P}$.

Antichains and names for reals 2

A $\mathbb{P}$-name for a real $\dot{x}$ is represented by sequences of maximal antichains $\{p_{n,i} : n \in \omega\}$ and of numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that

$$p_{n,i} \Vdash_{\mathbb{P}} \dot{x}(i) = k_{n,i}$$

Antichains and names for reals 2

A $\mathbb{P}$-name for a real $\dot{x}$ is represented by sequences of maximal antichains $\{p_{n,i} : n \in \omega\}$ and of numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that

$$p_{n,i} \Vdash_{\mathbb{P}} \dot{x}(i) = k_{n,i}$$

Therefore: a $\mathbb{P}^\kappa/\mathcal{D}$-name $\dot{y}$ for a real is represented by sequences $\{[f_{n,i}] : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that the $\{f_{n,i}(\alpha) : n \in \omega\}$, $i \in \omega$, form maximal antichains in $\mathbb{P}$ for all α and

$$[f_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{y}(i) = k_{n,i}$$

Antichains and names for reals 2

A $\mathbb{P}$-name for a real $\dot{x}$ is represented by sequences of maximal antichains $\{p_{n,i} : n \in \omega\}$ and of numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that

$$p_{n,i} \Vdash_{\mathbb{P}} \dot{x}(i) = k_{n,i}$$

Therefore: a $\mathbb{P}^\kappa/\mathcal{D}$-name $\dot{y}$ for a real is represented by sequences $\{[f_{n,i}] : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that the $\{f_{n,i}(\alpha) : n \in \omega\}$, $i \in \omega$, form maximal antichains in $\mathbb{P}$ for all α and

$$[f_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{y}(i) = k_{n,i}$$

The $\{f_{n,i}(\alpha) : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, determine a $\mathbb{P}$-name $\dot{y}_\alpha$ for a real given by

$$f_{n,i}(\alpha) \Vdash_{\mathbb{P}} \dot{y}_\alpha(i) = k_{n,i}$$

Antichains and names for reals 2

A $\mathbb{P}$-name for a real $\dot{x}$ is represented by sequences of maximal antichains $\{p_{n,i} : n \in \omega\}$ and of numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that

$$p_{n,i} \Vdash_{\mathbb{P}} \dot{x}(i) = k_{n,i}$$

Therefore: a $\mathbb{P}^\kappa/\mathcal{D}$-name $\dot{y}$ for a real is represented by sequences $\{[f_{n,i}] : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that the $\{f_{n,i}(\alpha) : n \in \omega\}$, $i \in \omega$, form maximal antichains in $\mathbb{P}$ for all α and

$$[f_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{y}(i) = k_{n,i}$$

The $\{f_{n,i}(\alpha) : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, determine a $\mathbb{P}$-name $\dot{y}_\alpha$ for a real given by

$$f_{n,i}(\alpha) \Vdash_{\mathbb{P}} \dot{y}_\alpha(i) = k_{n,i}$$

Think of $\dot{y}$ as the *mean* or *average* of the $\dot{y}_\alpha$ and write $\dot{y} = (\dot{y}_\alpha : \alpha < \kappa)/\mathcal{D}$.

Ultrapowers and eventual dominance 1

Lemma (ultrapowers and eventual dominance)

(i) $\mathbb{P} \Vdash$ "$\mathfrak{b} = \mathfrak{d} = \kappa$ *iff* $\dot{\mathbb{Q}}$ *adds a dominating real*".

(ii) *If* $\mathbb{P} \Vdash \mathfrak{b} > \kappa$ *or* $\mathbb{P} \Vdash \mathfrak{d} < \kappa$, *then* $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ *is* ω^ω*-bounding*".

Ultrapowers and eventual dominance 1

Lemma (ultrapowers and eventual dominance)

(i) $\mathbb{P} \Vdash$ "$\mathfrak{b} = \mathfrak{d} = \kappa$ *iff* $\dot{\mathbb{Q}}$ *adds a dominating real*".

(ii) *If* $\mathbb{P} \Vdash \mathfrak{b} > \kappa$ *or* $\mathbb{P} \Vdash \mathfrak{d} < \kappa$, *then* $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ *is* ω^ω*-bounding*".

Proof: (i) Assume $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \kappa\}$ is a scale".
Put $\dot{x} = (\dot{x}_\alpha : \alpha < \kappa)/\mathcal{D}$.
Clearly $p \Vdash_{\mathbb{P}\star\dot{\mathbb{Q}}} \dot{x} \geq^* \dot{x}_\alpha$ for all α.

Ultrapowers and eventual dominance 1

Lemma (ultrapowers and eventual dominance)

(i) $\mathbb{P} \Vdash$ "$\mathfrak{b} = \mathfrak{d} = \kappa$ *iff* $\dot{\mathbb{Q}}$ *adds a dominating real*".

(ii) *If* $\mathbb{P} \Vdash \mathfrak{b} > \kappa$ *or* $\mathbb{P} \Vdash \mathfrak{d} < \kappa$, *then* $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ *is* ω^ω*-bounding*".

Proof: (i) Assume $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \kappa\}$ is a scale".
Put $\dot{x} = (\dot{x}_\alpha : \alpha < \kappa)/\mathcal{D}$.
Clearly $p \Vdash_{\mathbb{P} \star \dot{\mathbb{Q}}} \dot{x} \geq^* \dot{x}_\alpha$ for all α.

Converse: exercise!

Ultrapowers and eventual dominance 1

Lemma (ultrapowers and eventual dominance)

(i) $\mathbb{P} \Vdash$ "$\mathfrak{b} = \mathfrak{d} = \kappa$ *iff* $\dot{\mathbb{Q}}$ *adds a dominating real*".

(ii) *If* $\mathbb{P} \Vdash \mathfrak{b} > \kappa$ *or* $\mathbb{P} \Vdash \mathfrak{d} < \kappa$, *then* $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ *is* ω^ω*-bounding*".

Proof: (i) Assume $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \kappa\}$ is a scale".
Put $\dot{x} = (\dot{x}_\alpha : \alpha < \kappa)/\mathcal{D}$.
Clearly $p \Vdash_{\mathbb{P}\star\dot{\mathbb{Q}}} \dot{x} \geq^* \dot{x}_\alpha$ for all α.

Converse: exercise!

(ii) Assume that $p \Vdash_{\mathbb{P}} \mathfrak{b} > \kappa$.
Let $\dot{y} = (\dot{y}_\alpha : \alpha < \kappa)/\mathcal{D}$ be a $\mathbb{P}^\kappa/\mathcal{D}$-name for a real.
The $\dot{y}_\alpha$ are forced to be bounded, say, by $\dot{x}$.
But then $p \Vdash_{\mathbb{P}\star\dot{\mathbb{Q}}} \dot{y} \leq^* \dot{x}$.

Ultrapowers and eventual dominance 2

Assume that for some $\mu < \kappa$, $p \Vdash_{\mathbb{P}} \mathfrak{d} = \mu$.
Say: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating".
Then: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating". $\square$

Ultrapowers and eventual dominance 2

Assume that for some $\mu < \kappa$, $p \Vdash_{\mathbb{P}} \mathfrak{d} = \mu$.
Say: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating".
Then: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating". $\square$

Problem

Give an exact characterization of when $\dot{\mathbb{Q}}$ is forced to be ω^ω-bounding.

Ultrapowers and eventual dominance 2

Assume that for some $\mu < \kappa$, $p \Vdash_{\mathbb{P}} \mathfrak{d} = \mu$.
Say: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating".
Then: $p \Vdash_{\mathbb{P}}$ "$\{\dot{x}_\alpha : \alpha < \mu\}$ is dominating". $\square$

Problem

Give an exact characterization of when $\dot{\mathbb{Q}}$ is forced to be ω^ω-bounding.

Main point: If $\mu > \kappa$ regular, and $\mathbb{P}$ forces $\mathfrak{b} = \mathfrak{d} = \mu$, this is preserved by taking ultrapowers.

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let* $\mu > \kappa$ *regular. Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *is* $\subseteq^*$*-decreasing and generates an ultrafilter". Then* $\mathbb{P}^\kappa / \mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *still generates an ultrafilter".*

(ii) *Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *satisfy* $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ *for* $\gamma < \delta$". *Then* $\mathbb{P}^\kappa / \mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *does not generate an ultrafilter".*

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let $\mu > \kappa$ regular. Assume $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, is $\subseteq^*$-decreasing and generates an ultrafilter". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, still generates an ultrafilter".*

(ii) *Assume $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, satisfy $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ for $\gamma < \delta$". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, does not generate an ultrafilter".*

Proof: (i) $\dot{B} = (\dot{B}_\alpha : \alpha < \kappa)/\mathcal{D}$: $\mathbb{P}^\kappa/\mathcal{D}$-name for a subset of ω.
By ccc: for each α, find $\gamma = \gamma_\alpha$ such that

$$\mathbb{P} \Vdash \text{"}\dot{A}_\gamma \subseteq^* \dot{B}_\alpha \text{ or } \dot{A}_\gamma \subseteq^* \omega \setminus \dot{B}_\alpha\text{"}. \quad (\star)$$

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let* $\mu > \kappa$ *regular. Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *is* $\subseteq^*$*-decreasing and generates an ultrafilter". Then* $\mathbb{P}^\kappa / \mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *still generates an ultrafilter".*

(ii) *Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *satisfy* $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ *for* $\gamma < \delta$". *Then* $\mathbb{P}^\kappa / \mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *does not generate an ultrafilter".*

Proof: (i) $\dot{B} = (\dot{B}_\alpha : \alpha < \kappa)/\mathcal{D}$: $\mathbb{P}^\kappa/\mathcal{D}$-name for a subset of ω.
By ccc: for each α, find $\gamma = \gamma_\alpha$ such that

$$\mathbb{P} \Vdash \text{“}\dot{A}_\gamma \subseteq^* \dot{B}_\alpha \text{ or } \dot{A}_\gamma \subseteq^* \omega \setminus \dot{B}_\alpha\text{”}. \quad (\star)$$

Let $\gamma = sup_\alpha \gamma_\alpha$. Then $(\star)$ holds for all α. Hence:

$$\mathbb{P}^\kappa/\mathcal{D} \Vdash \text{“}\dot{A}_\gamma \subseteq^* \dot{B} \text{ or } \dot{A}_\gamma \subseteq^* \omega \setminus \dot{B}\text{”}.$$

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let* $\mu > \kappa$ *regular. Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *is* $\subseteq^*$*-decreasing and generates an ultrafilter". Then* $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *still generates an ultrafilter".*

(ii) *Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *satisfy* $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ *for* $\gamma < \delta$". *Then* $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *does not generate an ultrafilter".*

Proof: (i) $\dot{B} = (\dot{B}_\alpha : \alpha < \kappa)/\mathcal{D}$: $\mathbb{P}^\kappa/\mathcal{D}$-name for a subset of ω.
By ccc: for each α, find $\gamma = \gamma_\alpha$ such that

$$\mathbb{P} \Vdash \text{“}\dot{A}_\gamma \subseteq^* \dot{B}_\alpha \text{ or } \dot{A}_\gamma \subseteq^* \omega \setminus \dot{B}_\alpha\text{”}. \quad (\star)$$

Let $\gamma = sup_\alpha \gamma_\alpha$. Then $(\star)$ holds for all α. Hence:

$$\mathbb{P}^\kappa/\mathcal{D} \Vdash \text{“}\dot{A}_\gamma \subseteq^* \dot{B} \text{ or } \dot{A}_\gamma \subseteq^* \omega \setminus \dot{B}\text{”}.$$

(ii) Exercise! (Consider $\dot{A} = (\dot{A}_\alpha : \alpha < \kappa)/\mathcal{D}$.) □

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let $\mu > \kappa$ regular. Assume $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, is $\subseteq^*$-decreasing and generates an ultrafilter". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, still generates an ultrafilter".*

(ii) *Assume $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, satisfy $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ for $\gamma < \delta$". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, does not generate an ultrafilter".*

Main points: (i) If $\mu > \kappa$ regular, and $\mathbb{P}$ forces an ultrafilter generated by a decreasing chain of length μ, this is preserved by taking ultrapowers.

Ultrapowers and ultrafilters

Lemma (ultrapowers and ultrafilters)

(i) *Let* $\mu > \kappa$ *regular. Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *is* $\subseteq^*$*-decreasing and generates an ultrafilter". Then* $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \mu$, *still generates an ultrafilter".*

(ii) *Assume* $\mathbb{P} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *satisfy* $\dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$ *for* $\gamma < \delta$". *Then* $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}_\gamma, \gamma < \kappa$, *does not generate an ultrafilter".*

Main points: (i) If $\mu > \kappa$ regular, and $\mathbb{P}$ forces an ultrafilter generated by a decreasing chain of length μ, this is preserved by taking ultrapowers.
(ii) Taking ultrapowers kills ultrafilter bases of size κ.

Ultrapowers and mad families

Lemma (ultrapowers and mad families)

Assume $\mathbb{P} \Vdash$ "$\dot{A}$ *is an a.d. family of size* $\geq \kappa$".
Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{A}$ *is not maximal*".
In particular, if $\mathbb{P}$ *forces* $\mathfrak{a} \geq \kappa$, *then no a.d. family of* $V^{\mathbb{P}}$ *is maximal in* $V^{\mathbb{P}^\kappa/\mathcal{D}}$.

Ultrapowers and mad families

Lemma (ultrapowers and mad families)

Assume $\mathbb{P} \Vdash$ "$\dot{\mathcal{A}}$ is an a.d. family of size $\geq \kappa$". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{\mathcal{A}}$ is not maximal". In particular, if $\mathbb{P}$ forces $\mathfrak{a} \geq \kappa$, then no a.d. family of $V^{\mathbb{P}}$ is maximal in $V^{\mathbb{P}^\kappa/\mathcal{D}}$.

Proof: Let $\mu \geq \kappa$. Let $\dot{\mathcal{A}} = \{\dot{A}_\gamma : \gamma < \mu\}$ be a $\mathbb{P}$-name for an a.d. family. Consider $\dot{A} = (\dot{A}_\alpha : \alpha < \kappa)/\mathcal{D}$.

Ultrapowers and mad families

Lemma (ultrapowers and mad families)

Assume $\mathbb{P} \Vdash$ "$\dot{\mathcal{A}}$ is an a.d. family of size $\geq \kappa$". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{\mathcal{A}}$ is not maximal". In particular, if $\mathbb{P}$ forces $\mathfrak{a} \geq \kappa$, then no a.d. family of $V^{\mathbb{P}}$ is maximal in $V^{\mathbb{P}^\kappa/\mathcal{D}}$.

Proof: Let $\mu \geq \kappa$. Let $\dot{\mathcal{A}} = \{\dot{A}_\gamma : \gamma < \mu\}$ be a $\mathbb{P}$-name for an a.d. family. Consider $\dot{A} = (\dot{A}_\alpha : \alpha < \kappa)/\mathcal{D}$.

Claim: $\dot{A}$ is forced to be a.d. from all members of $\dot{\mathcal{A}}$.

Ultrapowers and mad families

Lemma (ultrapowers and mad families)

Assume $\mathbb{P} \Vdash$ "$\dot{\mathcal{A}}$ is an a.d. family of size $\geq \kappa$". Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{\mathcal{A}}$ is not maximal". In particular, if $\mathbb{P}$ forces $\mathfrak{a} \geq \kappa$, then no a.d. family of $V^{\mathbb{P}}$ is maximal in $V^{\mathbb{P}^\kappa/\mathcal{D}}$.

Proof: Let $\mu \geq \kappa$. Let $\dot{\mathcal{A}} = \{\dot{A}_\gamma : \gamma < \mu\}$ be a $\mathbb{P}$-name for an a.d. family. Consider $\dot{A} = (\dot{A}_\alpha : \alpha < \kappa)/\mathcal{D}$.

Claim: $\dot{A}$ is forced to be a.d. from all members of $\dot{\mathcal{A}}$.

Fix $\gamma < \mu$. For $\alpha < \kappa$ with $\alpha \neq \gamma$: $\Vdash_{\mathbb{P}} |\dot{A}_\gamma \cap \dot{A}_\alpha| < \omega$
Thus: $\{\alpha < \kappa : \Vdash_{\mathbb{P}} |\dot{A}_\gamma \cap \dot{A}_\alpha| < \omega\}$ belongs to $\mathcal{D}$.
Hence: $\Vdash_{\mathbb{P}^\kappa/\mathcal{D}} |\dot{A}_\gamma \cap \dot{A}| < \omega$. $\square$

Ultrapowers and mad families

Lemma (ultrapowers and mad families)

Assume $\mathbb{P} \Vdash$ "$\dot{\mathcal{A}}$ *is an a.d. family of size* $\geq \kappa$".
Then $\mathbb{P}^\kappa/\mathcal{D} \Vdash$ "$\dot{\mathcal{A}}$ *is not maximal*".
In particular, if $\mathbb{P}$ *forces* $\mathfrak{a} \geq \kappa$, *then no a.d. family of* $V^{\mathbb{P}}$ *is maximal in* $V^{\mathbb{P}^\kappa/\mathcal{D}}$.

Main point: Taking ultrapowers kills mad families of size $\geq \kappa$.

Preservation of complete embeddability

We next look at ultrapowers of whole iterations.
The basic result says:

Preservation of complete embeddability

We next look at ultrapowers of whole iterations.
The basic result says:

Lemma (Preservation of complete embeddability)

Assume $\mathbb{P} \lessdot \mathbb{Q}$. *Then* $\mathbb{P}^\kappa/\mathcal{D} \lessdot \mathbb{Q}^\kappa/\mathcal{D}$.

Preservation of complete embeddability

We next look at ultrapowers of whole iterations.
The basic result says:

Lemma (Preservation of complete embeddability)

Assume $\mathbb{P} <\circ \mathbb{Q}$. *Then* $\mathbb{P}^\kappa / \mathcal{D} <\circ \mathbb{Q}^\kappa / \mathcal{D}$.

Proof: By elementarity:

Preservation of complete embeddability

We next look at ultrapowers of whole iterations.
The basic result says:

Lemma (Preservation of complete embeddability)

Assume $\mathbb{P} \lessdot \mathbb{Q}$. *Then* $\mathbb{P}^\kappa/\mathcal{D} \lessdot \mathbb{Q}^\kappa/\mathcal{D}$.

Proof: By elementarity:
Assume D predense in $\mathbb{P}^\kappa/\mathcal{D}$.
Then: $\{\alpha < \kappa : \{f(\alpha) : [f] \in D\}$ predense in $\mathbb{P}\} \in \mathcal{D}$.
Hence: $\{\alpha < \kappa : \{f(\alpha) : [f] \in D\}$ predense in $\mathbb{Q}\} \in \mathcal{D}$.
Thus: D predense in $\mathbb{Q}^\kappa/\mathcal{D}$. $\square$

Ultrapowers of iterations

Assume $(\mathbb{P}_\gamma : \gamma \leq \mu)$ is an iteration.
Then: $(\mathbb{P}^\kappa_\gamma/\mathcal{D} : \gamma \leq \mu)$ is again an iteration.

Ultrapowers of iterations

Assume $(\mathbb{P}_\gamma : \gamma \leq \mu)$ is an iteration.
Then: $(\mathbb{P}_\gamma^\kappa / \mathcal{D} : \gamma \leq \mu)$ is again an iteration.
Note that we make no requirements about limits.
In fact, "being a direct limit" is in general NOT preserved by taking the ultrapower:

Ultrapowers of iterations

Assume $(\mathbb{P}_\gamma : \gamma \leq \mu)$ is an iteration.
Then: $(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma \leq \mu)$ is again an iteration.
Note that we make no requirements about limits.
In fact, "being a direct limit" is in general NOT preserved by taking the ultrapower:

Lemma (Ultrapower of an iteration)

Assume $\mathbb{P}_\mu = \lim\operatorname{dir}(\mathbb{P}_\gamma : \gamma < \mu)$.
Then $\lim\operatorname{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu) \lessdot \mathbb{P}_\mu^\kappa/\mathcal{D}$.
Also $\mathbb{P}_\mu^\kappa/\mathcal{D} = \lim\operatorname{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu)$ *iff* $cf(\mu) \neq \kappa$.

Ultrapowers of iterations

Lemma (Ultrapower of an iteration)

Assume $\mathbb{P}_\mu = \lim \mathrm{dir}(\mathbb{P}_\gamma : \gamma < \mu)$.
Then $\lim \mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu) \lessdot \mathbb{P}_\mu^\kappa/\mathcal{D}$.
Also $\mathbb{P}_\mu^\kappa/\mathcal{D} = \lim \mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu)$ *iff* $cf(\mu) \neq \kappa$.

Proof: Second statement: Let $[f] \in \mathbb{P}_\mu^\kappa/\mathcal{D}$.

Ultrapowers of iterations

Lemma (Ultrapower of an iteration)

Assume $\mathbb{P}_\mu = \lim\mathrm{dir}(\mathbb{P}_\gamma : \gamma < \mu)$.
Then $\lim\mathrm{dir}(\mathbb{P}^\kappa_\gamma/\mathcal{D} : \gamma < \mu) \lessdot \mathbb{P}^\kappa_\mu/\mathcal{D}$.
Also $\mathbb{P}^\kappa_\mu/\mathcal{D} = \lim\mathrm{dir}(\mathbb{P}^\kappa_\gamma/\mathcal{D} : \gamma < \mu)$ *iff* $cf(\mu) \neq \kappa$.

Proof: Second statement: Let $[f] \in \mathbb{P}^\kappa_\mu/\mathcal{D}$.

$cf(\mu) \neq \kappa$: there is $\gamma < \mu$ such that $\{\alpha : f(\alpha) \in \mathbb{P}_\gamma\} \in \mathcal{D}$.
Hence: $[f] \in \mathbb{P}^\kappa_\gamma/\mathcal{D}$.
Therefore: $\mathbb{P}^\kappa_\mu/\mathcal{D}$ is direct limit.

Ultrapowers of iterations

Lemma (Ultrapower of an iteration)

Assume $\mathbb{P}_\mu = \lim\mathrm{dir}(\mathbb{P}_\gamma : \gamma < \mu)$.
Then $\lim\mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu) \lessdot \mathbb{P}_\mu^\kappa/\mathcal{D}$.
Also $\mathbb{P}_\mu^\kappa/\mathcal{D} = \lim\mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu)$ *iff* $cf(\mu) \neq \kappa$.

Proof: Second statement: Let $[f] \in \mathbb{P}_\mu^\kappa/\mathcal{D}$.

$cf(\mu) \neq \kappa$: there is $\gamma < \mu$ such that $\{\alpha : f(\alpha) \in \mathbb{P}_\gamma\} \in \mathcal{D}$.
Hence: $[f] \in \mathbb{P}_\gamma^\kappa/\mathcal{D}$.
Therefore: $\mathbb{P}_\mu^\kappa/\mathcal{D}$ is direct limit.

$cf(\mu) = \kappa$ and $(\gamma_\alpha : \alpha < \kappa)$ is cofinal in μ:
choose $f \in \mathbb{P}_\mu^\kappa$ with $f(\alpha) \in \mathbb{P}_\mu \setminus \mathbb{P}_{\gamma_\alpha}$.
Then $[f] \in \mathbb{P}_\mu^\kappa/\mathcal{D}$ does not belong to the direct limit.

Ultrapowers of iterations

Lemma (Ultrapower of an iteration)

Assume $\mathbb{P}_\mu = \lim\mathrm{dir}(\mathbb{P}_\gamma : \gamma < \mu)$.
Then $\lim\mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu) <\circ \mathbb{P}_\mu^\kappa/\mathcal{D}$.
Also $\mathbb{P}_\mu^\kappa/\mathcal{D} = \lim\mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu)$ *iff* $cf(\mu) \neq \kappa$.

Proof:

First statement: assume $cf(\mu) > \omega$.
Assume $\{[f_n] : n \in \omega\}$ maximal antichain in $\lim\mathrm{dir}(\mathbb{P}_\gamma^\kappa/\mathcal{D} : \gamma < \mu)$.
Then: $\{[f_n] : n \in \omega\}$ maximal antichain in some $\mathbb{P}_\gamma^\kappa/\mathcal{D}$.
Therefore, also maximal in $\mathbb{P}_\mu^\kappa/\mathcal{D}$. $\square$

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.
(That is,

- $\mathbb{D}_{\gamma+1} = \mathbb{D}_\gamma \star \dot{\mathbb{D}}$
- $\mathbb{D}_\delta = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{D}_\gamma$ for limit δ.)

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.

Obtain iteration $(\mathbb{D}_\gamma^\kappa / \mathcal{D} : \gamma \leq \mu)$ such that:

- $\mathbb{D}_\delta^\kappa / \mathcal{D} = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{D}_\gamma^\kappa / \mathcal{D}$ iff $cf(\delta) \neq \kappa$
  (In particular, this is true for $\delta = \mu$.)

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.

Obtain iteration $(\mathbb{D}^\kappa_\gamma/\mathcal{D} : \gamma \leq \mu)$ such that:

- $\mathbb{D}^\kappa_\delta/\mathcal{D} = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{D}^\kappa_\gamma/\mathcal{D}$ iff $cf(\delta) \neq \kappa$
- $\mathbb{D}^\kappa_{\gamma+1}/\mathcal{D} = \mathbb{D}^\kappa_\gamma/\mathcal{D} \star \dot{\mathbb{D}}$

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.

Obtain iteration $(\mathbb{D}^\kappa_\gamma / \mathcal{D} : \gamma \leq \mu)$ such that:

- $\mathbb{D}^\kappa_\delta / \mathcal{D} = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{D}^\kappa_\gamma / \mathcal{D}$ iff $cf(\delta) \neq \kappa$
- $\mathbb{D}^\kappa_{\gamma+1} / \mathcal{D} = \mathbb{D}^\kappa_\gamma / \mathcal{D} \star \dot{\mathbb{D}}$
- $(\mathbb{D}^\kappa_\gamma / \mathcal{D} : \gamma < \mu)$ is an fsi of Hechler forcing of length $j(\mu)$ (I.e. $\mathbb{D}^\kappa_\gamma / \mathcal{D} = \mathbb{D}_{j(\gamma)}$.)

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.

Obtain iteration $(\mathbb{D}^\kappa_\gamma/\mathcal{D} : \gamma \leq \mu)$ such that:

- $\mathbb{D}^\kappa_\delta/\mathcal{D} = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{D}^\kappa_\gamma/\mathcal{D}$ iff $cf(\delta) \neq \kappa$
- $\mathbb{D}^\kappa_{\gamma+1}/\mathcal{D} = \mathbb{D}^\kappa_\gamma/\mathcal{D} \star \dot{\mathbb{D}}$
- $(\mathbb{D}^\kappa_\gamma/\mathcal{D} : \gamma < \mu)$ is an fsi of Hechler forcing of length $j(\mu)$
- The dominating family added by $\mathbb{D}_\mu$ is still dominating in $V^{\mathbb{D}^\kappa_\mu/\mathcal{D}}$

Example for ultrapower of an iteration

Let us look at an example of an iteration and its ultrapower.

Fix regular $\mu > \kappa$.
Let $(\mathbb{D}_\gamma : \gamma \leq \mu)$ be the fsi of Hechler forcing $\mathbb{D}$.

Obtain iteration $(\mathbb{D}^\kappa_\gamma/\mathcal{D} : \gamma \leq \mu)$ such that:

- $\mathbb{D}^\kappa_\delta/\mathcal{D} = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{D}^\kappa_\gamma/\mathcal{D}$ iff $cf(\delta) \neq \kappa$
- $\mathbb{D}^\kappa_{\gamma+1}/\mathcal{D} = \mathbb{D}^\kappa_\gamma/\mathcal{D} \star \dot{\mathbb{D}}$
- $(\mathbb{D}^\kappa_\gamma/\mathcal{D} : \gamma < \mu)$ is an fsi of Hechler forcing of length $j(\mu)$
- The dominating family added by $\mathbb{D}_\mu$ is still dominating in $V^{\mathbb{D}^\kappa_\mu/\mathcal{D}}$
- No a.d. family of $V^{\mathbb{D}_\mu}$ is mad in $V^{\mathbb{D}^\kappa_\mu/\mathcal{D}}$

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^2_\gamma := (\mathbb{P}^1_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^2_\gamma : \gamma \leq \mu)$. Etc.

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^2_\gamma := (\mathbb{P}^1_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^2_\gamma : \gamma \leq \mu)$. Etc.

More generally, for $\alpha < \lambda$,
put $\mathbb{P}^{\alpha+1}_\gamma := (\mathbb{P}^\alpha_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^{\alpha+1}_\gamma : \gamma \leq \mu)$.

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^2_\gamma := (\mathbb{P}^1_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^2_\gamma : \gamma \leq \mu)$. Etc.

More generally, for $\alpha < \lambda$,
put $\mathbb{P}^{\alpha+1}_\gamma := (\mathbb{P}^\alpha_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^{\alpha+1}_\gamma : \gamma \leq \mu)$.

What do we do for limit α?

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^2_\gamma := (\mathbb{P}^1_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^2_\gamma : \gamma \leq \mu)$. Etc.

More generally, for $\alpha < \lambda$,
put $\mathbb{P}^{\alpha+1}_\gamma := (\mathbb{P}^\alpha_\gamma)^\kappa / \mathcal{D}$. Obtain iteration $(\mathbb{P}^{\alpha+1}_\gamma : \gamma \leq \mu)$.

What do we do for limit α?
For some applications $\mathbb{P}^\alpha_\gamma = \lim\mathrm{dir}_{\beta<\alpha}\mathbb{P}^\beta_\gamma$ will be OK.

Matrices of iterated ultrapowers

Assume $\lambda > \mu > \kappa$ regular.

Start with iteration $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$. Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$.
Put $\mathbb{P}^2_\gamma := (\mathbb{P}^1_\gamma)^\kappa/\mathcal{D}$. Obtain iteration $(\mathbb{P}^2_\gamma : \gamma \leq \mu)$. Etc.

More generally, for $\alpha < \lambda$,
put $\mathbb{P}^{\alpha+1}_\gamma := (\mathbb{P}^\alpha_\gamma)^\kappa/\mathcal{D}$. Obtain iteration $(\mathbb{P}^{\alpha+1}_\gamma : \gamma \leq \mu)$.

What do we do for limit α?
For some applications $\mathbb{P}^\alpha_\gamma = \lim\operatorname{dir}_{\beta<\alpha}\mathbb{P}^\beta_\gamma$ will be OK.
For some applications want something else:
Suppose $(\mathbb{D}^\beta_\gamma : \gamma \leq \mu)$ are such that $\mathbb{D}^\beta_{\gamma+1} = \mathbb{D}^\beta_\gamma \star \dot{\mathbb{D}}$ for $\beta < \alpha$.
Then still want $\mathbb{D}^\alpha_{\gamma+1} = \mathbb{D}^\alpha_\gamma \star \dot{\mathbb{D}}$. Doable but more complicated!

More cardinal invariants

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal
(I.e., for all $C \in [\omega]^\omega$ there is $A \in \mathcal{A}$ with $|C \cap A| = \omega$.)

More cardinal invariants

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal

$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A}$ is infinite mad$\}$, the *almost disjointness number*.

More cardinal invariants

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal

$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A}$ is infinite mad$\}$, the *almost disjointness number*.

$\mathcal{U}$ ultrafilter on ω.
$\mathcal{F}$ *base* of $\mathcal{U}$: for all $A \in \mathcal{U}$ there is $B \in \mathcal{F}$ with $B \subseteq^* A$.

More cardinal invariants

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal

$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A}$ is infinite mad$\}$, the *almost disjointness number*.

$\mathcal{U}$ ultrafilter on ω.
$\mathcal{F}$ *base* of $\mathcal{U}$: for all $A \in \mathcal{U}$ there is $B \in \mathcal{F}$ with $B \subseteq^* A$.

$\chi(\mathcal{U}) := \min\{|\mathcal{F}| : \mathcal{F}$ base of $\mathcal{U}\}$, the *character of* $\mathcal{U}$.
$\mathfrak{u} := \min\{\chi(\mathcal{U}) : \mathcal{U}$ ultrafilter on $\omega\}$, the *ultrafilter number*.

More cardinal invariants

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal

$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A}$ is infinite mad$\}$, the *almost disjointness number*.

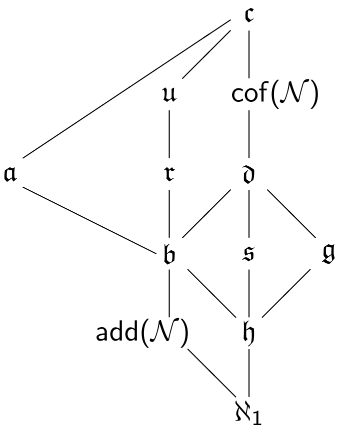
$\mathcal{U}$ ultrafilter on ω.
$\mathcal{F}$ *base* of $\mathcal{U}$: for all $A \in \mathcal{U}$ there is $B \in \mathcal{F}$ with $B \subseteq^* A$.

$\chi(\mathcal{U}) := \min\{|\mathcal{F}| : \mathcal{F}$ base of $\mathcal{U}\}$, the *character of* $\mathcal{U}$.
$\mathfrak{u} := \min\{\chi(\mathcal{U}) : \mathcal{U}$ ultrafilter on $\omega\}$, the *ultrafilter number*.

Theorem

(i) $\mathfrak{b} \leq \mathfrak{a}$
(ii) $\mathfrak{r} \leq \mathfrak{u}$

ZFC-inequalities: another diagram



First application: $\mathfrak{a}$ versus $\mathfrak{d}$

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mu$ holds.
In particular $\mathfrak{d} < \mathfrak{a}$ is consistent.

First application: $\mathfrak{a}$ versus $\mathfrak{d}$

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mu$ holds.

In particular $\mathfrak{d} < \mathfrak{a}$ is consistent.

Proof: Start with $(\mathbb{D}^0_\gamma : \gamma \leq \mu)$: fsi of Hechler forcing.
Repeatedly take ultrapower to get $\mathbb{D}^{\alpha+1}_\gamma = (\mathbb{D}^\alpha_\gamma)^\kappa / \mathcal{D}$.
Guarantee in limit step α that still $\mathbb{D}^\alpha_{\gamma+1} = \mathbb{D}^\alpha_\gamma \star \dot{\mathbb{D}}$.

First application: $\mathfrak{a}$ versus $\mathfrak{d}$

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mu$ holds.
In particular $\mathfrak{d} < \mathfrak{a}$ is consistent.

Proof: Start with $(\mathbb{D}^0_\gamma : \gamma \leq \mu)$: fsi of Hechler forcing.
Repeatedly take ultrapower to get $\mathbb{D}^{\alpha+1}_\gamma = (\mathbb{D}^\alpha_\gamma)^\kappa / \mathcal{D}$.
Guarantee in limit step α that still $\mathbb{D}^\alpha_{\gamma+1} = \mathbb{D}^\alpha_\gamma \star \dot{\mathbb{D}}$.

$\mathfrak{a} \geq \lambda$: small a.d. families destroyed by ultrapower.

First application: $\mathfrak{a}$ versus $\mathfrak{d}$

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mu$ holds.
In particular $\mathfrak{d} < \mathfrak{a}$ is consistent.

Proof: Start with $(\mathbb{D}^0_\gamma : \gamma \leq \mu)$: fsi of Hechler forcing.
Repeatedly take ultrapower to get $\mathbb{D}^{\alpha+1}_\gamma = (\mathbb{D}^\alpha_\gamma)^\kappa / \mathcal{D}$.
Guarantee in limit step α that still $\mathbb{D}^\alpha_{\gamma+1} = \mathbb{D}^\alpha_\gamma \star \dot{\mathbb{D}}$.

$\mathfrak{a} \geq \lambda$: small a.d. families destroyed by ultrapower.

$\mathfrak{b} = \mathfrak{d} = \mu$: $(\mathbb{D}^\lambda_\gamma : \gamma \leq \mu)$ still iteration of $\mathbb{D}$ (though not with direct limits). $\square$

First application: $\mathfrak{a}$ versus $\mathfrak{d}$

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular.
Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mu$ holds.
In particular $\mathfrak{d} < \mathfrak{a}$ is consistent.

Remark: Using iterations along templates, Shelah also proved $CON(\mathfrak{d} < \mathfrak{a})$ on the basis of $CON(ZFC)$ alone.

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

(ii) $\mathbb{P}^0_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$-generic"

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

(ii) $\mathbb{P}^0_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$-generic"

(iii) $\mathbb{P}^0_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^0_\gamma$ for $\delta < \gamma$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular.
Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

(ii) $\mathbb{P}^0_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$-generic"

(iii) $\mathbb{P}^0_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^0_\gamma$ for $\delta < \gamma$

(iv) $\mathbb{P}^0_{\gamma+1} = \mathbb{P}^0_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

(ii) $\mathbb{P}^0_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$-generic"

(iii) $\mathbb{P}^0_\gamma \Vdash \operatorname{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^0_\gamma$ for $\delta < \gamma$

(iv) $\mathbb{P}^0_{\gamma+1} = \mathbb{P}^0_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$

Note: (iii) implies

(v) $\mathbb{P}^0_{\gamma+1} \Vdash \dot{\mathcal{U}}^0_\delta \subseteq \dot{\mathcal{U}}^0_\gamma$ and $\operatorname{ran}(\dot{\ell}_\gamma) \subseteq^* \operatorname{ran}(\dot{\ell}_\delta)$ for $\delta < \gamma$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 1

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mu$ holds.
In particular $\mathfrak{u} < \mathfrak{a}$ is consistent.

Proof: Build fsi $(\mathbb{P}^0_\gamma : \gamma \leq \mu)$ and names $(\dot{\mathcal{U}}^0_\gamma : \gamma \leq \mu)$, $(\dot{\ell}_\gamma : \gamma < \mu)$ such that

(i) $\mathbb{P}^0_\gamma \Vdash \dot{\mathcal{U}}^0_\gamma$ is an ultrafilter

(ii) $\mathbb{P}^0_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$-generic"

(iii) $\mathbb{P}^0_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^0_\gamma$ for $\delta < \gamma$

(iv) $\mathbb{P}^0_{\gamma+1} = \mathbb{P}^0_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^0_\gamma}$

Hence: $\mathbb{P}^0_\mu$ forces $\dot{\mathcal{U}}^0_\mu$ is generated by $\mathrm{ran}(\dot{\ell}_\gamma)$, $\gamma < \mu$.

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim\mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

(v) $\mathbb{P}^1_{\gamma+1} = \mathbb{P}^1_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

(v) $\mathbb{P}^1_{\gamma+1} = \mathbb{P}^1_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$

(vi) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\delta \subseteq \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim\mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

(v) $\mathbb{P}^1_{\gamma+1} = \mathbb{P}^1_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$

(vi) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\delta \subseteq \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

Repeat this to get $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^\kappa/\mathcal{D}$.
Guarantee in limit step α that still $\mathbb{P}^\alpha_{\gamma+1} = \mathbb{P}^\alpha_\gamma \star \dot{\mathbb{L}}_{\dot{\mathcal{U}}^\alpha_\gamma}$.

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa / \mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

(v) $\mathbb{P}^1_{\gamma+1} = \mathbb{P}^1_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$

(vi) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\delta \subseteq \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

Repeat this to get $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^\kappa / \mathcal{D}$.
Guarantee in limit step α that still $\mathbb{P}^\alpha_{\gamma+1} = \mathbb{P}^\alpha_\gamma \star \dot{\mathbb{L}}_{\dot{\mathcal{U}}^\alpha_\gamma}$.

$\mathfrak{a} \geq \lambda$: small a.d. families destroyed by ultrapower.

Second application: $\mathfrak{a}$ versus $\mathfrak{u}$ 2

Take the ultrapower $\mathbb{P}^1_\gamma := (\mathbb{P}^0_\gamma)^\kappa/\mathcal{D}$.
Obtain iteration $(\mathbb{P}^1_\gamma : \gamma \leq \mu)$ such that:

(i) $\mathbb{P}^1_\delta = \lim \mathrm{dir}_{\gamma<\delta}\mathbb{P}^1_\gamma$ iff $cf(\delta) \neq \kappa$

(ii) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\gamma$ is an ultrafilter extending $\dot{\mathcal{U}}^0_\gamma$

(iii) $\mathbb{P}^1_\gamma \Vdash$ "$\dot{\ell}_\gamma$ is the name for the $\mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$-generic"

(iv) $\mathbb{P}^1_\gamma \Vdash \mathrm{ran}(\dot{\ell}_\delta) \in \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

(v) $\mathbb{P}^1_{\gamma+1} = \mathbb{P}^1_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^1_\gamma}$

(vi) $\mathbb{P}^1_\gamma \Vdash \dot{\mathcal{U}}^1_\delta \subseteq \dot{\mathcal{U}}^1_\gamma$ for $\delta < \gamma$

Repeat this to get $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^\kappa/\mathcal{D}$.
Guarantee in limit step α that still $\mathbb{P}^\alpha_{\gamma+1} = \mathbb{P}^\alpha_\gamma \star \dot{\mathbb{L}}_{\dot{\mathcal{U}}^\alpha_\gamma}$.

$\mathfrak{u} = \mu$: taking ultrapowers preserves ultrafilters generated by chains of length μ. □

Third application: character spectrum

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \kappa$ is regular. Then there is a ccc forcing extension in which $\mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \aleph_1$ holds, and there is no ultrafilter of character κ. In particular it is consistent that the character spectrum is non-convex.

Third application: character spectrum

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \kappa$ is regular. Then there is a ccc forcing extension in which $\mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \aleph_1$ holds, and there is no ultrafilter of character κ. In particular it is consistent that the character spectrum is non-convex.

Proof sketch: As in previous proof with μ replaced by $\aleph_1$ and $\mathbb{P}_0^0$ adds at least κ Cohen reals.
(This guarantees the ultrapowers are nontrivial.)

Third application: character spectrum

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \kappa$ is regular. Then there is a ccc forcing extension in which $\mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \aleph_1$ holds, and there is no ultrafilter of character κ. In particular it is consistent that the character spectrum is non-convex.

Proof sketch: As in previous proof with μ replaced by $\aleph_1$ and $\mathbb{P}^0_0$ adds at least κ Cohen reals.

κ not character: taking ultrapowers kills ultrafilter bases of size κ.

Third application: character spectrum

Theorem (Shelah)

Assume κ is measurable, and $\lambda = \lambda^\omega > \kappa$ is regular. Then there is a ccc forcing extension in which $\mathfrak{c} = \lambda$ and $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \aleph_1$ holds, and there is no ultrafilter of character κ. In particular it is consistent that the character spectrum is non-convex.

Proof sketch: As in previous proof with μ replaced by $\aleph_1$ and $\mathbb{P}^0_0$ adds at least κ Cohen reals.

κ not character: taking ultrapowers kills ultrafilter bases of size κ.

$\mathfrak{u} = \aleph_1$ (and thus character): as before.
$\mathfrak{c} = \lambda$ character: in *ZFC*. □

Forth application: $\mathfrak{a}$ and $\mathfrak{s}$ versus $\mathfrak{b}$

Theorem (B.-Fischer)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mu$ holds.

Forth application: $\mathfrak{a}$ and $\mathfrak{s}$ versus $\mathfrak{b}$

Theorem (B.-Fischer)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mu$ holds.

Proof sketch: $\mathbb{P}^0_\gamma$ adds γ Cohen reals, $\gamma \leq \mu$.

Forth application: $\mathfrak{a}$ and $\mathfrak{s}$ versus $\mathfrak{b}$

Theorem (B.-Fischer)

Assume κ is measurable, and $\lambda = \lambda^\omega > \mu > \kappa$ are regular. Then there is a ccc forcing extension in which $\mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \lambda$ and $\mathfrak{b} = \mu$ holds.

Proof sketch: $\mathbb{P}^0_\gamma$ adds γ Cohen reals, $\gamma \leq \mu$.
Combine the methods of lectures 2 and 3 to make $\mathfrak{s}$ and $\mathfrak{a}$ large while keeping $\mathfrak{b}$ small.
Build fsi $(\mathbb{P}^\alpha_\gamma : \alpha \leq \lambda)$ such that

(i) for even α, $\mathbb{P}^{\alpha+1}_\gamma = \mathbb{P}^\alpha_\gamma \star \mathbb{M}_{\dot{\mathcal{U}}^\alpha_\gamma}$

(ii) for odd α, $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^\kappa / \mathcal{D}$ $\square$

Relatives of $\mathfrak{g}$ and $\mathfrak{h}$

Today we look at $\mathfrak{g}$ and $\mathfrak{h}$ and their relatives.
Suslin ccc iterations and matrix iterations of lectures 1 through 3 keep these cardinals small.
So such iterations cannot be used to separate them.

Relatives of $\mathfrak{g}$ and $\mathfrak{h}$

Today we look at $\mathfrak{g}$ and $\mathfrak{h}$ and their relatives.
Suslin ccc iterations and matrix iterations of lectures 1 through 3 keep these cardinals small.
So such iterations cannot be used to separate them.

To separate two such cardinals, we need to build a small witness for one *along* the iteration while killing all small witnesses for the other.

Relatives of $\mathfrak{g}$ and $\mathfrak{h}$

Today we look at $\mathfrak{g}$ and $\mathfrak{h}$ and their relatives.
Suslin ccc iterations and matrix iterations of lectures 1 through 3 keep these cardinals small.
So such iterations cannot be used to separate them.

To separate two such cardinals, we need to build a small witness for one *along* the iteration while killing all small witnesses for the other.

For the latter task, use a diamond principle.

$\mathfrak{g}$ and $\mathfrak{g}_f$ 1

Recall:

A family $\mathcal{D} \subseteq [\omega]^\omega$ is *groupwise dense* if

- $\mathcal{D}$ is open
  $(\forall A \in \mathcal{D} \;\; \forall B \subseteq^* A \;\; (B \in \mathcal{D}))$
- given a partition $(I_n : n \in \omega)$ of ω into intervals, there is $B \in [\omega]^\omega$ such that $\bigcup_{n \in B} I_n \in \mathcal{D}$
  (this implies, in particular, that $\mathcal{D}$ is dense)

$\mathfrak{g}$ and $\mathfrak{g}_f$ 1

Recall:

A family $\mathcal{D} \subseteq [\omega]^\omega$ is *groupwise dense* if

- $\mathcal{D}$ is open
  $(\forall A \in \mathcal{D} \;\; \forall B \subseteq^* A \;\; (B \in \mathcal{D}))$
- given a partition $(I_n : n \in \omega)$ of ω into intervals, there is $B \in [\omega]^\omega$ such that $\bigcup_{n \in B} I_n \in \mathcal{D}$
  (this implies, in particular, that $\mathcal{D}$ is dense)

$\mathcal{D}$ is a *groupwise dense ideal* if it is groupwise dense and closed under finite unions.

Remark: $\mathcal{D}$ groupwise dense ideal $\iff$ dual filter $\mathcal{D}^*$ non-meager.

$\mathfrak{g}$ and $\mathfrak{g}_f$ 2

$\mathfrak{g} := \min\{|\mathfrak{D}| :$ all $\mathcal{D} \in \mathfrak{D}$ groupwise dense and $\bigcap \mathfrak{D} = \emptyset\}$
the *groupwise density number*.
$\mathfrak{g}_f := \min\{|\mathfrak{D}| :$ all $\mathcal{D} \in \mathfrak{D}$ groupwise dense ideals and $\bigcap \mathfrak{D} = \emptyset\}$
the *groupwise density number for ideals*.

$\mathfrak{g}$ and $\mathfrak{g}_f$ 2

$\mathfrak{g} := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ groupwise dense and } \bigcap \mathfrak{D} = \emptyset\}$
the *groupwise density number*.
$\mathfrak{g}_f := \min\{|\mathfrak{D}| : \text{all } \mathcal{D} \in \mathfrak{D} \text{ groupwise dense ideals and } \bigcap \mathfrak{D} = \emptyset\}$
the *groupwise density number for ideals*.

Clearly $\mathfrak{g} \leq \mathfrak{g}_f$. We show:

Theorem (B.)

$CON(\mathfrak{g} < \mathfrak{g}_f)$.

Context: filter dichotomy and semifilter trichotomy

filter dichotomy FD: $\forall$ filters $\mathcal{F}$ on ω, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{F})$ is the cofinite filter or $f(\mathcal{F})$ is an ultrafilter.

Context: filter dichotomy and semifilter trichotomy

filter dichotomy FD: $\forall$ filters $\mathcal{F}$ on ω, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{F})$ is the cofinite filter or $f(\mathcal{F})$ is an ultrafilter.
semi-filter trichotomy: $\forall$ families $\mathcal{X} \subseteq [\omega]^\omega$ closed under almost supersets, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{X})$ is the cofinite filter or $f(\mathcal{X}) = [\omega]^\omega$ or $f(\mathcal{X})$ is an ultrafilter.

Context: filter dichotomy and semifilter trichotomy

filter dichotomy FD: $\forall$ filters $\mathcal{F}$ on ω, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{F})$ is the cofinite filter or $f(\mathcal{F})$ is an ultrafilter.
semi-filter trichotomy: $\forall$ families $\mathcal{X} \subseteq [\omega]^\omega$ closed under almost supersets, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{X})$ is the cofinite filter or $f(\mathcal{X}) = [\omega]^\omega$ or $f(\mathcal{X})$ is an ultrafilter.

Theorem (Blass-Laflamme)

(i) *filter dichotomy FD is equivalent to* $\mathfrak{u} < \mathfrak{g}_f$

(ii) *semi-filter trichotomy is equivalent to* $\mathfrak{u} < \mathfrak{g}$

Context: filter dichotomy and semifilter trichotomy

filter dichotomy FD: $\forall$ filters $\mathcal{F}$ on ω, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{F})$ is the cofinite filter or $f(\mathcal{F})$ is an ultrafilter.
semi-filter trichotomy: $\forall$ families $\mathcal{X} \subseteq [\omega]^\omega$ closed under almost supersets, $\exists f : \omega \to \omega$ finite-to-one such that either $f(\mathcal{X})$ is the cofinite filter or $f(\mathcal{X}) = [\omega]^\omega$ or $f(\mathcal{X})$ is an ultrafilter.

Theorem (Blass-Laflamme)

(i) *filter dichotomy FD is equivalent to* $\mathfrak{u} < \mathfrak{g}_f$
(ii) *semi-filter trichotomy is equivalent to* $\mathfrak{u} < \mathfrak{g}$

Question (Blass)

Are filter dichotomy and semi-filter trichotomy equivalent?

In our model for $\mathfrak{g} < \mathfrak{g}_f$: $\mathfrak{u} = \mathfrak{g}_f$.

Outline of proof

Theorem (B.)

$CON(\mathfrak{g} < \mathfrak{g}_f)$.

Outline of proof:

Outline of proof

Theorem (B.)

CON$(\mathfrak{g} < \mathfrak{g}_f)$.

Outline of proof:

Assume *CH* and build fsi of ccc partial orders of length ω_2.
Along the iteration also build a witness $\mathfrak{D}$ for $\mathfrak{g} = \aleph_1$.
Use a diamond principle to kill (initial segments of) potential witnesses $\mathfrak{E}$ for $\mathfrak{g}_f = \aleph_1$ in limit stages of cofinality ω_1.

Outline of proof

Theorem (B.)

$CON(\mathfrak{g} < \mathfrak{g}_f)$.

Outline of proof:

Assume CH and build fsi of ccc partial orders of length ω_2.
Along the iteration also build a witness $\mathfrak{D}$ for $\mathfrak{g} = \aleph_1$.
Use a diamond principle to kill (initial segments of) potential witnesses $\mathfrak{E}$ for $\mathfrak{g}_f = \aleph_1$ in limit stages of cofinality ω_1.
The main point is that in such a limit stage a certain filter can be built such that Laver forcing with this filter kills $\mathfrak{E}$ while at the same time not destroying (the initial part of) $\mathfrak{D}$
(see Crucial Lemma below).

The forcing

$\diamondsuit_{S^2_1}$: there is a sequence $(S_\alpha \subseteq \alpha : \alpha < \omega_2$ and $cf(\alpha) = \omega_1)$ such that $\forall S \subseteq \omega_2$ $\exists$ stationarily many α with $S \cap \alpha = S_\alpha$.

The forcing

$\diamondsuit_{S^2_1}$: there is a sequence $(S_\alpha \subseteq \alpha : \alpha < \omega_2$ and $cf(\alpha) = \omega_1)$ such that $\forall S \subseteq \omega_2$ $\exists$ stationarily many α with $S \cap \alpha = S_\alpha$.

Build fsi $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2)$ of ccc forcing such that

(i) if $cf(\alpha) = \omega_1$, then $\dot{\mathbb{Q}}_\alpha = \mathbb{L}_{\dot{\mathcal{F}}_\alpha}$
(see below for details)

The forcing

$\diamondsuit_{S^2_1}$: there is a sequence $(S_\alpha \subseteq \alpha : \alpha < \omega_2$ and $cf(\alpha) = \omega_1)$ such that $\forall S \subseteq \omega_2$ $\exists$ stationarily many α with $S \cap \alpha = S_\alpha$.

Build fsi $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2)$ of ccc forcing such that

(i) if $cf(\alpha) = \omega_1$, then $\dot{\mathbb{Q}}_\alpha = \mathbb{L}_{\dot{\mathcal{F}}_\alpha}$
(see below for details)

(ii) if $cf(\alpha) \leq \omega$, then $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{D}}$

Building witnesses 1

Construct groupwise dense families $\mathcal{D}_\beta$, $\beta < \omega_1$, along the iteration to witness $\mathfrak{g} = \aleph_1$.
Require $\mathcal{D}_{\beta'} \subseteq \mathcal{D}_\beta$ for $\beta' \geq \beta$.

Building witnesses 1

Construct groupwise dense families $\mathcal{D}_\beta$, $\beta < \omega_1$, along the iteration to witness $\mathfrak{g} = \aleph_1$.
Require $\mathcal{D}_{\beta'} \subseteq \mathcal{D}_\beta$ for $\beta' \geq \beta$.

More explicitly: have $\mathcal{D}_\beta^{\leq\alpha} = \mathcal{D}_\beta \cap V_\alpha$ such that

- $\mathcal{D}_{\beta'}^{\leq\alpha} \subseteq \mathcal{D}_\beta^{\leq\alpha}$ for $\beta' \geq \beta$

Building witnesses 1

Construct groupwise dense families $\mathcal{D}_\beta$, $\beta < \omega_1$, along the iteration to witness $\mathfrak{g} = \aleph_1$.
Require $\mathcal{D}_{\beta'} \subseteq \mathcal{D}_\beta$ for $\beta' \geq \beta$.

More explicitly: have $\mathcal{D}_\beta^{\leq\alpha} = \mathcal{D}_\beta \cap V_\alpha$ such that

- $\mathcal{D}_{\beta'}^{\leq\alpha} \subseteq \mathcal{D}_\beta^{\leq\alpha}$ for $\beta' \geq \beta$
- $\mathcal{D}_\beta^{\leq\alpha}$ open
  (but not necessarily groupwise dense)

Building witnesses 1

Construct groupwise dense families $\mathcal{D}_\beta$, $\beta < \omega_1$, along the iteration to witness $\mathfrak{g} = \aleph_1$.
Require $\mathcal{D}_{\beta'} \subseteq \mathcal{D}_\beta$ for $\beta' \geq \beta$.

More explicitly: have $\mathcal{D}_\beta^{\leq\alpha} = \mathcal{D}_\beta \cap V_\alpha$ such that

- $\mathcal{D}_{\beta'}^{\leq\alpha} \subseteq \mathcal{D}_\beta^{\leq\alpha}$ for $\beta' \geq \beta$
- $\mathcal{D}_\beta^{\leq\alpha}$ open
- additional conditions, guaranteeing $\mathcal{D}_\beta$ will be groupwise dense

Building witnesses 2

To show that $\bigcap_{\beta<\omega_1} \mathcal{D}_\beta = \emptyset$, need

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \exists \beta < \omega_1 \quad A \notin \mathcal{D}_\beta \tag{$+_\alpha$}$$

Building witnesses 2

To show that $\bigcap_{\beta<\omega_1} \mathcal{D}_\beta = \emptyset$, need

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \exists \beta < \omega_1 \quad A \notin \mathcal{D}_\beta \tag{$+_\alpha$}$$

Argue that

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \exists \beta < \omega_1 \quad A \notin \mathcal{D}_\beta^{\leq\alpha} \tag{$*_\alpha$}$$

and

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \forall \beta < \omega_1 \quad (A \notin \mathcal{D}_\beta^{\leq\alpha} \text{ implies } A \notin \mathcal{D}_\beta^{\leq\alpha+1}) \tag{$\dagger_\alpha$}$$

Building witnesses 2

To show that $\bigcap_{\beta<\omega_1} \mathcal{D}_\beta = \emptyset$, need

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \exists \beta < \omega_1 \quad A \notin \mathcal{D}_\beta \qquad (+_\alpha)$$

Argue that

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \exists \beta < \omega_1 \quad A \notin \mathcal{D}_\beta^{\leq\alpha} \qquad (*_\alpha)$$

and

$$\forall A \in [\omega]^\omega \cap V_\alpha \quad \forall \beta < \omega_1 \quad (A \notin \mathcal{D}_\beta^{\leq\alpha} \text{ implies } A \notin \mathcal{D}_\beta^{\leq\alpha+1}) \quad (\dagger_\alpha)$$

Straightforward: $(+_\alpha)$ follows from $(*_\alpha)$ and $(\dagger_\alpha)$.
Easy: $(\dagger_\alpha)$ holds.
Main point: proof of $(*_\alpha)$ by induction on α.
Standard: $(*_\alpha)$ for α limit and $\alpha = \alpha' + 1$, $cf(\alpha') \leq \omega$.

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.
Also construct filter $\mathcal{F}_\alpha$ such that forcing with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{F}_\alpha}$ over V_α destroys potential witness for $\mathfrak{g}_f = \aleph_1$.

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.
Also construct filter $\mathcal{F}_\alpha$ such that forcing with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{F}_\alpha}$ over V_α destroys potential witness for $\mathfrak{g}_f = \aleph_1$.
We want:

(i) if $\mathcal{E}_\beta, \beta < \omega_1$, is the initial segment of a potential witness for $\mathfrak{g}_f = \aleph_1$, handed down by $\diamondsuit_{S^2_1}$, then $\mathcal{F}_\alpha$ diagonalizes the $\mathcal{E}_\beta$ (that is, for all $\beta < \omega_1$, $\mathcal{F}_\alpha \cap \mathcal{E}_\beta \neq \emptyset$)

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.
Also construct filter $\mathcal{F}_\alpha$ such that forcing with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{F}_\alpha}$ over V_α destroys potential witness for $\mathfrak{g}_f = \aleph_1$.
We want:

(i) if $\mathcal{E}_\beta, \beta < \omega_1$, is the initial segment of a potential witness for $\mathfrak{g}_f = \aleph_1$, handed down by $\diamondsuit_{S_1^2}$, then $\mathcal{F}_\alpha$ diagonalizes the $\mathcal{E}_\beta$ (that is, for all $\beta < \omega_1$, $\mathcal{F}_\alpha \cap \mathcal{E}_\beta \neq \emptyset$)

(ii) for all partial functions $f : \omega \to \omega$ from V_α with $\mathrm{dom}(f) \in \mathcal{F}_\alpha^+$ and $f^{-1}(n) \notin \mathcal{F}_\alpha^+$ for all $n \in \omega$, there is $\beta < \omega_1$ such that for all $F \in \mathcal{F}_\alpha$, $f(F \cap \mathrm{dom}(f)) \notin \mathcal{D}_\beta^{\leq\alpha}$

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.
Also construct filter $\mathcal{F}_\alpha$ such that forcing with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{F}_\alpha}$ over V_α destroys potential witness for $\mathfrak{g}_f = \aleph_1$.
We want:

(i) if $\mathcal{E}_\beta, \beta < \omega_1$, is the initial segment of a potential witness for $\mathfrak{g}_f = \aleph_1$, handed down by $\diamondsuit_{S^2_1}$, then $\mathcal{F}_\alpha$ diagonalizes the $\mathcal{E}_\beta$ (that is, for all $\beta < \omega_1$, $\mathcal{F}_\alpha \cap \mathcal{E}_\beta \neq \emptyset$)

(ii) for all partial functions $f : \omega \to \omega$ from V_α with $\mathrm{dom}(f) \in \mathcal{F}_\alpha^+$ and $f^{-1}(n) \notin \mathcal{F}_\alpha^+$ for all $n \in \omega$, there is $\beta < \omega_1$ such that for all $F \in \mathcal{F}_\alpha$, $f(F \cap \mathrm{dom}(f)) \notin \mathcal{D}_\beta^{\leq\alpha}$

(i): for destroying a witness of $\mathfrak{g}_f = \aleph_1$.
(ii): for proving $(*_{\alpha+1})$ (and thus building a witness for $\mathfrak{g} = \aleph_1$).

Building and destroying witnesses 1

Main issue: proof of $(*_{\alpha+1})$ in case $cf(\alpha) = \omega_1$.
Also construct filter $\mathcal{F}_\alpha$ such that forcing with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{F}_\alpha}$ over V_α destroys potential witness for $\mathfrak{g}_f = \aleph_1$.
We want:

(i) if $\mathcal{E}_\beta, \beta < \omega_1$, is the initial segment of a potential witness for $\mathfrak{g}_f = \aleph_1$, handed down by $\diamondsuit_{S^2_1}$, then $\mathcal{F}_\alpha$ diagonalizes the $\mathcal{E}_\beta$ (that is, for all $\beta < \omega_1$, $\mathcal{F}_\alpha \cap \mathcal{E}_\beta \neq \emptyset$)

(ii) for all partial functions $f : \omega \to \omega$ from V_α with $\mathrm{dom}(f) \in \mathcal{F}_\alpha^+$ and $f^{-1}(n) \notin \mathcal{F}_\alpha^+$ for all $n \in \omega$, there is $\beta < \omega_1$ such that for all $F \in \mathcal{F}_\alpha$, $f(F \cap \mathrm{dom}(f)) \notin \mathcal{D}_\beta^{\leq\alpha}$

(i): for destroying a witness of $\mathfrak{g}_f = \aleph_1$.
(ii): for proving $(*_{\alpha+1})$ (and thus building a witness for $\mathfrak{g} = \aleph_1$).

Crucial Lemma

Assume $(*_\alpha)$. In V_α, there is $\mathcal{F}_\alpha$ satisfying (i) and (ii) above.

Building and destroying witnesses 2

Crucial Corollary

Assume $cf(\alpha) = \omega_1$ *and* $(*_\alpha)$ *holds. Then* $(*_{\alpha+1})$ *is true as well.*

Building and destroying witnesses 2

Crucial Corollary

Assume cf$(\alpha) = \omega_1$ and $(*_\alpha)$ holds. Then $(*_{\alpha+1})$ is true as well.

Proof:

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names:

Building and destroying witnesses 2

Crucial Corollary

Assume $cf(\alpha) = \omega_1$ *and* $(*_\alpha)$ *holds. Then* $(*_{\alpha+1})$ *is true as well.*

Proof:

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names:

φ: statement of the forcing language.

σ *forces* φ: $\exists p \in \mathbb{L}_{\mathcal{F}}$ with $\mathrm{stem}(p) = \sigma$ and $p \Vdash \varphi$.

Building and destroying witnesses 2

Crucial Corollary

Assume $cf(\alpha) = \omega_1$ and $(*_\alpha)$ holds. Then $(*_{\alpha+1})$ is true as well.

Proof:

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names:

φ: statement of the forcing language.

σ *forces* φ: $\exists p \in \mathbb{L}_{\mathcal{F}}$ with $\mathrm{stem}(p) = \sigma$ and $p \Vdash \varphi$.

$\rho_\varphi(\sigma) = 0$ if σ forces φ.

$\alpha > 0$: $\rho_\varphi(\sigma) \leq \alpha$ if $\{n : \rho_\varphi(\sigma^\frown n) < \alpha\} \in \mathcal{F}^+$.

Building and destroying witnesses 2

Crucial Corollary

Assume $cf(\alpha) = \omega_1$ *and* $(*_\alpha)$ *holds. Then* $(*_{\alpha+1})$ *is true as well.*

Proof:

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names:

φ: statement of the forcing language.

σ *forces* φ: $\exists p \in \mathbb{L}_{\mathcal{F}}$ with $\mathrm{stem}(p) = \sigma$ and $p \Vdash \varphi$.

$\rho_\varphi(\sigma) = 0$ if σ forces φ.

$\alpha > 0$: $\rho_\varphi(\sigma) \leq \alpha$ if $\{n : \rho_\varphi(\sigma^\frown n) < \alpha\} \in \mathcal{F}^+$.

σ *favors* φ if $\rho_\varphi(\sigma)$ is defined (i.e., it is less than ω_1).

σ forces at most one of φ and $\neg\varphi$ and favors at least one of them.

In fact, σ favors φ iff σ does not force $\neg\varphi$.

Building and destroying witnesses 3

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names, continued:

Let $\dot{A}$ be an $\mathbb{L}_{\mathcal{F}}$-name for an infinite subset of ω.

Building and destroying witnesses 3

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names, continued:

Let $\dot{A}$ be an $\mathbb{L}_{\mathcal{F}}$-name for an infinite subset of ω.

$rk(\sigma) = 0$ if

- either there is $B \in [\omega]^\omega$ such that, for all $n \in B$, σ favors $n \in \dot{A}$
- or there is a partial function $f : \omega \to \omega$ such that $\mathrm{dom}(f) \in \mathcal{F}^+$, $f^{-1}(n) \notin \mathcal{F}^+$ for all $n \in \omega$, and $\sigma^\frown n$ favors $f(n) \in \dot{A}$ for all $n \in \mathrm{dom}(f)$

Building and destroying witnesses 3

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names, continued:

Let $\dot{A}$ be an $\mathbb{L}_{\mathcal{F}}$-name for an infinite subset of ω.

$rk(\sigma) = 0$ if

- either there is $B \in [\omega]^\omega$ such that, for all $n \in B$, σ favors $n \in \dot{A}$
- or there is a partial function $f : \omega \to \omega$ such that $\mathrm{dom}(f) \in \mathcal{F}^+$, $f^{-1}(n) \notin \mathcal{F}^+$ for all $n \in \omega$, and $\sigma^\frown n$ favors $f(n) \in \dot{A}$ for all $n \in \mathrm{dom}(f)$

$\alpha > 0$: $rk(\sigma) \leq \alpha$ if $\{n : rk(\sigma^\frown n) < \alpha\} \in \mathcal{F}^+$.

Building and destroying witnesses 3

Rank analysis of $\mathbb{L}_{\mathcal{F}_\alpha}$-names, continued:

Let $\dot{A}$ be an $\mathbb{L}_{\mathcal{F}}$-name for an infinite subset of ω.

$rk(\sigma) = 0$ if

- either there is $B \in [\omega]^\omega$ such that, for all $n \in B$, σ favors $n \in \dot{A}$
- or there is a partial function $f : \omega \to \omega$ such that $\mathrm{dom}(f) \in \mathcal{F}^+$, $f^{-1}(n) \notin \mathcal{F}^+$ for all $n \in \omega$, and $\sigma^\frown n$ favors $f(n) \in \dot{A}$ for all $n \in \mathrm{dom}(f)$

$\alpha > 0$: $rk(\sigma) \leq \alpha$ if $\{n : rk(\sigma^\frown n) < \alpha\} \in \mathcal{F}^+$.

Claim: $rk(\sigma)$ is defined for all σ. $\square$

Building and destroying witnesses 4

For σ with $rk(\sigma) = 0$ fix either a witness B_σ or a witness f_σ as in the definition of rk.

Building and destroying witnesses 4

For σ with $rk(\sigma) = 0$ fix either a witness B_σ or a witness f_σ as in the definition of rk.
For σ of rank 0 such that B_σ is defined, use $(*_\alpha)$ to find γ_σ such that $B_\sigma \notin \mathcal{D}^{\leq\alpha}_{\gamma_\sigma}$.

Building and destroying witnesses 4

For σ with $rk(\sigma) = 0$ fix either a witness B_σ or a witness f_σ as in the definition of rk.
For σ of rank 0 such that B_σ is defined, use $(*_\alpha)$ to find γ_σ such that $B_\sigma \notin \mathcal{D}^{\leq\alpha}_{\gamma_\sigma}$.
For σ of rank 0 such that f_σ is defined, use property (ii), which holds for $\mathcal{F}_\alpha$ by Crucial Lemma, to find γ_σ such that for all $F \in \mathcal{F}_\alpha$, $f_\sigma(F \cap \mathrm{dom}(f_\sigma)) \notin \mathcal{D}^{\leq\alpha}_{\gamma_\sigma}$.

Building and destroying witnesses 4

For σ with $rk(\sigma) = 0$ fix either a witness B_σ or a witness f_σ as in the definition of rk.

For σ of rank 0 such that B_σ is defined, use $(*_\alpha)$ to find γ_σ such that $B_\sigma \notin \mathcal{D}^{\leq\alpha}_{\gamma_\sigma}$.

For σ of rank 0 such that f_σ is defined, use property (ii), which holds for $\mathcal{F}_\alpha$ by Crucial Lemma, to find γ_σ such that for all $F \in \mathcal{F}_\alpha$, $f_\sigma(F \cap \mathrm{dom}(f_\sigma)) \notin \mathcal{D}^{\leq\alpha}_{\gamma_\sigma}$.

Let $\beta \geq \sup_\sigma \gamma_\sigma$.

Building and destroying witnesses 5

Claim: $\Vdash \dot{A} \notin \mathcal{D}_{\beta}^{\leq\alpha+1}$.

Building and destroying witnesses 5

Claim: $\Vdash \dot{A} \notin \mathcal{D}_{\beta}^{\leq\alpha+1}$.

Assume: $\exists B \in \mathcal{D}_{\beta}^{\leq\alpha}$ and $p \in \mathbb{L}_{\mathcal{F}_\alpha}$ such that $p \Vdash \dot{A} \subseteq B$.
Wlog: $\sigma := \mathrm{stem}(p)$ has rank 0.

Building and destroying witnesses 5

Claim: $\Vdash \dot{A} \notin \mathcal{D}_\beta^{\leq\alpha+1}$.

Assume: $\exists B \in \mathcal{D}_\beta^{\leq\alpha}$ and $p \in \mathbb{L}_{\mathcal{F}_\alpha}$ such that $p \Vdash \dot{A} \subseteq B$.
Wlog: $\sigma := \mathrm{stem}(p)$ has rank 0.

Assume first B_σ is defined. By assumption: $B_\sigma \setminus B$ is infinite. Choose $k \in B_\sigma \setminus B$. Since σ favors $k \in \dot{A}$: $\exists q \leq p$ such that $q \Vdash k \in \dot{A}$, a contradiction.

Building and destroying witnesses 5

Claim: $\Vdash \dot{A} \notin \mathcal{D}_\beta^{\leq\alpha+1}$.

Assume: $\exists B \in \mathcal{D}_\beta^{\leq\alpha}$ and $p \in \mathbb{L}_{\mathcal{F}_\alpha}$ such that $p \Vdash \dot{A} \subseteq B$.
Wlog: $\sigma := \mathrm{stem}(p)$ has rank 0.

Assume first B_σ is defined. By assumption: $B_\sigma \setminus B$ is infinite. Choose $k \in B_\sigma \setminus B$. Since σ favors $k \in \dot{A}$: $\exists q \leq p$ such that $q \Vdash k \in \dot{A}$, a contradiction.

Assume next f_σ is defined. Let $F := \mathrm{succ}_p(\sigma)$. By (ii): $f_\sigma(F \cap \mathrm{dom}(f_\sigma)) \notin \mathcal{D}_\beta^{\leq\alpha}$. Hence: choose $n \in F \cap \mathrm{dom}(f_\sigma)$ such that $k := f_\sigma(n) \notin B$. Since $\sigma^\frown n$ favors $k \in \dot{A}$: $\exists q \leq p$ with $\mathrm{stem}(q) \supseteq \sigma^\frown n$ such that $q \Vdash k \in \dot{A}$, again a contradiction.

Building and destroying witnesses 5

Claim: $\Vdash \dot{A} \notin \mathcal{D}_\beta^{\leq \alpha+1}$.

Assume: $\exists B \in \mathcal{D}_\beta^{\leq\alpha}$ and $p \in \mathbb{L}_{\mathcal{F}_\alpha}$ such that $p \Vdash \dot{A} \subseteq B$.
Wlog: $\sigma := \mathrm{stem}(p)$ has rank 0.

Assume first B_σ is defined. By assumption: $B_\sigma \setminus B$ is infinite. Choose $k \in B_\sigma \setminus B$. Since σ favors $k \in \dot{A}$: $\exists q \leq p$ such that $q \Vdash k \in \dot{A}$, a contradiction.

Assume next f_σ is defined. Let $F := \mathrm{succ}_p(\sigma)$. By (ii): $f_\sigma(F \cap \mathrm{dom}(f_\sigma)) \notin \mathcal{D}_\beta^{\leq\alpha}$. Hence: choose $n \in F \cap \mathrm{dom}(f_\sigma)$ such that $k := f_\sigma(n) \notin B$. Since $\sigma^\frown n$ favors $k \in \dot{A}$: $\exists q \leq p$ with $\mathrm{stem}(q) \supseteq \sigma^\frown n$ such that $q \Vdash k \in \dot{A}$, again a contradiction.

Proves Crucial Corollary. $\square$

End of proof

Corollary

$\mathfrak{g} = \aleph_1$ *holds in* V_{ω_2}

End of proof

Corollary

$\mathfrak{g} = \aleph_1$ *holds in* V_{ω_2}

Proof: Know: $(*_\alpha)$ holds for all α. Implies: $\mathfrak{g} = \aleph_1$. $\square$

End of proof

Corollary

$\mathfrak{g} = \aleph_1$ *holds in* V_{ω_2}

Corollary

$\mathfrak{g}_f = \aleph_2$ *holds in* V_{ω_2}

Proof: $\mathfrak{E} = \{\mathcal{E}_\beta : \beta < \omega_1\}$ family of groupwise dense ideals.
By $\diamondsuit_{S^2_1}$ and (i) of Crucial Lemma:
$\exists \alpha$ such that $(\mathcal{E}_\beta \cap V_\alpha) \cap \mathcal{F}_\alpha \neq \emptyset$ for all $\beta < \omega_1$.

End of proof

Corollary

$\mathfrak{g} = \aleph_1$ *holds in* V_{ω_2}

Corollary

$\mathfrak{g}_f = \aleph_2$ *holds in* V_{ω_2}

Proof: $\mathfrak{E} = \{\mathcal{E}_\beta : \beta < \omega_1\}$ family of groupwise dense ideals.
By $\diamondsuit_{S^2_1}$ and (i) of Crucial Lemma:
$\exists\alpha$ such that $(\mathcal{E}_\beta \cap V_\alpha) \cap \mathcal{F}_\alpha \neq \emptyset$ for all $\beta < \omega_1$.

$\mathbb{L}_{\mathcal{F}_\alpha}$ adds pseudointersection through filter $\mathcal{F}_\alpha$, i.e., a set $X \in [\omega]^\omega$ such that for all $\beta < \omega_1$ there is $B_\beta \in \mathcal{E}_\beta \cap V_\alpha$ with $X \subseteq^* B_\beta$.
$\mathcal{E}_\beta$ open: $X \in \bigcap_\beta \mathcal{E}_\beta$. Thus $\mathfrak{E}$ cannot witness $\mathfrak{g}_f = \aleph_1$. $\square$